

Gaussian matrix model in an external field and non-intersecting Brownian motions

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Abstract: We study the Gaussian hermitian random matrix ensemble with an external matrix which has an arbitrary number of eigenvalues with arbitrary multiplicity. We compute the limiting eigenvalues correlations when the size of the matrix goes to infinity in non-critical regimes. We show that they exhibit universal behavior and can be expressed with the Sine and Airy kernels. We also briefly review the implication of these results in terms of non-intersecting Brownian motions.

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1 Introduction and statement results

In these notes, we are interested in the hermitian matrix model in an external field defined by the partition function:

$$Z := \int_{H_N} dM e^{-N \text{Tr} \left(\frac{M^2}{2} - AM \right)} \quad (1-1)$$

where one integrates over the hermitian matrices M of size $N \times N$, A is a deterministic diagonal matrix of the form

$$A := \text{diag}(\overbrace{a_1, \dots, a_1}^{n_1}, \overbrace{a_2, \dots, a_2}^{n_2}, \dots, \overbrace{a_k, \dots, a_k}^{n_k}) \quad (1-2)$$

with $a_i \neq a_j$ for $i \neq j$ and the measure dM is the product of the Lebesgue measure of the real entries of the matrix M :

$$dM := \prod_{i < j} d\text{Re}(M_{ij}) d\text{Im}(M_{ij}) \prod_i dM_{ii}. \quad (1-3)$$

The main motivation of this work comes from the study of the exclusion process defined as follows. Consider N independent non-intersecting Brownian motions $\{x_i(t)\}_{i=1}^N$

starting from $x = 0$ at time $t = 0$ and ending at point $x = a_i(1)$ at $t = 1$ for n_i of them with $i = 1, \dots, k$. One obviously imposes $\sum_{i=1}^k n_i = N$. Indeed, it is known [10, 2] that, up to a rescaling,

$$m_i = \frac{x_i}{t(1-t)}, \quad (1-4)$$

the position of the particles $x_i(t)$ at any time $t \in (0, 1)$, are distributed as the eigenvalues m_i of the random matrix M submitted to an external matrix $A(t)$ whose eigenvalues are given by

$$a_i(t) = a_i(1) \sqrt{\frac{t}{1-t}}. \quad (1-5)$$

We study the behavior of this process when the number N of Brownian motions gets very large, i.e. the size of the random matrix M gets large and the filling fractions $\frac{n_i}{N} = \epsilon_i$ are kept fixed². In this limit, at a given time t the Brownian motions accumulate to fill a set of $l(t)$ intervals with $1 \leq l(t) \leq k$ and the behavior near the boundaries: $l(t) \rightarrow 1$ as $t \rightarrow 0$ and $l(t) \rightarrow k$ as $t \rightarrow 1$. Indeed, at the beginning, all the particles start at the origin and stay next to $x = 0$ at short times. At the end, the particles are split into k groups, each composed of n_i particles, lying next to the k end points $a_i(1)$. As time grows, the unique interval corresponding to $t = 0$ splits into two intervals for a critical value of time corresponding to a phase transition. Keeping time growing, these intervals split also for critical values of time and this process keeps on iteratively until one gets k intervals reaching the end points. In these notes, one describes the distribution of Brownian particles in these intervals for any non critical value of time by studying the associated matrix model for different regimes. In section 3, we study the case of large time when $l(t) = k$ as a warm up before studying the generic non-critical case $1 \leq l(t) \leq k$ in section 4.

In both sections, we follow the same procedure. We first study the Pastur equation [9], or spectral curve,

$$\prod_{i=1}^k (y - a_i(t)) \left(x - y - \sum_{j=1}^k \frac{n_j}{N(y - a_j(t))} \right) = 0, \quad (1-6)$$

of the matrix model obtained for time t , as well as the associated Riemann surface. The structure of the Riemann surface changes with time: especially, it has as many cuts as the number l of intervals $[z_{2i-1}, z_{2i}]_{i=1}^l$ filled by the eigenvalues. We then study an associated Riemann-Hilbert problem of size $(k+1) \times (k+1)$ whose solution $Y(x)$

²In order to impose this condition, one has to restrict our study to N common multiple of the denominators of the fractions ϵ_i . It should be possible to release this constraint by showing that the corrections one should get by adding an isolated eigenvalue to A are not seen at the leading order of the physical quantities when $N \rightarrow \infty$. This issue is not essential for the present work and is not studied here.

allows to define a fundamental kernel $K_N(x, y)$ giving access to any correlation function of the eigenvalues through:

$$R_m(x_1, x_2, \dots, x_m) = \det (K_N(x_j, x_k))_{1 \leq j, k \leq m}. \quad (1-7)$$

By successive transformations of this RH problem, we turn it to a simple one exponentially close when $N \rightarrow \infty$ to a model RH problem whose solution can be written explicitly. This asymptotic solution of the RH problem leads to the main theorems of this paper. The first one describe the limiting density of Brownian motion at a given time when we have a large number of them.

Theorem 1.1 *The limiting density of eigenvalues*

$$\rho(x) := \lim_{N \rightarrow \infty} \frac{K_N(x, x)}{N} \quad (1-8)$$

exists, is supported by the real cuts $[z_{2i-1}, z_{2i}]_{i=1}^l$ and is expressed in terms of the unique solution $y_0(x)$ of the Pastur's equation

$$\prod_{i=1}^k (y - a_i(t)) \left(x - y - \sum_{j=1}^k \frac{n_j}{N(y - a_j(t))} \right) = 0 \quad (1-9)$$

such that

$$y_0(x) \sim \frac{1}{x} + O\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \rightarrow \infty \quad (1-10)$$

by

$$\rho(x) = \frac{1}{\pi} |\operatorname{Im} y_0(x)|. \quad (1-11)$$

It is analytic on the cuts and vanishes as a square root at the edges z_i .

The second one describes the behavior of the kernel inside the support of the eigenvalues (or Brownian motions).

Theorem 1.2 *For any $x \in \bigcup_{i=1}^l (z_{2i-1}, z_{2i})$ and any pair $(u, v) \in \mathbb{R}^2$, the kernel converges to the universal sine kernel:*

$$\lim_{N \rightarrow \infty} \frac{1}{N\rho(x)} \widehat{K}_N \left(x + \frac{u}{N\rho(x)}, x + \frac{v}{N\rho(x)} \right) = \frac{\sin \pi(u - v)}{\pi(u - v)}$$

(1-12)

where we use the rescaled kernel:

$$\widehat{K}_N(x, y) = e^{N(h(x) - h(y))} K_N(x, y) \quad (1-13)$$

with the function

$$h(x) := -\frac{x^2}{4} + \operatorname{Re} \left[\int^x y_0(x) \right] \quad (1-14)$$

where the primitive will be made precise later.

The last one describes the asymptotic behavior of the kernel at the edge of the support of eigenvalues z_i .

Theorem 1.3 *For any $i = 1, \dots, 2k$ and every $(u, v) \in \mathbb{R}^2$, the rescaled kernel is given by the universal Airy kernel on the edges of the supports of eigenvalues:*

$$\lim_{N \rightarrow \infty} \frac{1}{(\rho_i N)^{\frac{2}{3}}} \widehat{K}_N \left(z_i + (-1)^i \frac{u}{(\rho_i N)^{\frac{2}{3}}}, z_i + (-1)^i \frac{v}{(\rho_i N)^{\frac{2}{3}}} \right) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u - v}, \quad (1-15)$$

where ρ_i is a constant characterizing the behavior of ρ near the branch point z_i :

$$\rho(x) = \frac{\rho_i}{\pi} |x - z_i|^{\frac{1}{2}} (1 + o(1)) \quad (1-16)$$

when $x \rightarrow z_i$.

Organization of the paper:

- section 2 is a reminder on the Gaussian matrix model with an external field and its link to non-intersecting Brownian motions.
- section 3 is a warm up where we study the asymptotics of the kernel and mean density of eigenvalues in a particular non degenerate case: there are as many supports of eigenvalues as the number of different eigenvalues of the external matrix A : $l = k$.
- in section 4, we study the general case $1 \leq l \leq k$ and prove the theorems 1.1, 1.2 and 1.3 for any non-critical spectral curve Eq. (1-6).
- section 5 is a conclusion and present some possible generalizations of the present work.

2 Gaussian matrix integral in an external field, reminder

2.1 Definitions

In this section we consider the hermitian matrix model defined through the partition function

$$Z := \int_{H_N} dM e^{-N \text{Tr} \left(\frac{M^2}{2} - AM \right)} \quad (2-1)$$

where one integrates over the hermitian matrices M of size $N \times N$, A is a deterministic diagonal matrix of the form

$$A := \text{diag}(\overbrace{a_1, \dots, a_1}^{n_1}, \overbrace{a_2, \dots, a_2}^{n_2}, \dots, \overbrace{a_k, \dots, a_k}^{n_k}) \quad (2-2)$$

and the measure dM is the product of the Lebesgue measure of the real entries of the matrix M :

$$dM := \prod_{i < j} d\text{Re}(M_{ij}) d\text{Im}(M_{ij}) \prod_i dM_{ii}. \quad (2-3)$$

By diagonalizing the matrix M , we can reduce this problem to the study of its eigenvalues x_1, \dots, x_N , which are constrained by the probability measure:

$$d\mu(x_1, \dots, x_N) = \prod_i dx_i \frac{\Delta(x)}{\Delta(a_i)} e^{-N \sum_i \frac{x_i^2}{2}} \det(e^{-x_i a_j}) \quad (2-4)$$

where $\Delta(x)$ denotes the Vandermonde determinant and one considers the eigenvalues a_i with their multiplicities.

They can be seen as a Coulomb gas composed k groups of n_k particles living on the real axis and submitted to a gaussian potential $V_i(x) = \frac{x^2}{2} - a_i x$ and a logarithmic repulsive interaction $\ln|x_i - x_j|$. The position of the eigenvalues is then an equilibrium configuration for such particles. When the size of the matrix tends to infinity, the number of eigenvalues grows in the same way as the latter and they fill a finite set of intervals $[z_{2i-1}, z_{2i}]_{i=1}^l$ with $1 \leq l \leq k$. The number l of such intervals as well as their positions depend on the a_i 's and the ratios $\epsilon_i := \frac{n_i}{N}$. They can be encoded in an algebraic curve as we remind now.

Let us define the density of states as the one point correlation function:

$$\rho(x) := \frac{1}{N} \int_{\mathbb{R}^N} d\mu(u_1, u_2, \dots, u_N) \sum_{i=1}^N \delta(x - u_i), \quad (2-5)$$

as well as its Stieltjes transform, called the resolvent

$$W(x) := \int_{\mathbb{R}} \frac{\rho(u)}{x - u} du. \quad (2-6)$$

Remark that the density of states ρ is expected, by the analogy with a Coulomb gas, to have a compact support composed of the l intervals corresponding to the l saddles of the effective potential for the Coulomb particles. Thus, the resolvent can be written:

$$W(x) = \sum_{i=1}^l \frac{1}{N} \int_{[z_{2i-1}, z_{2i}]} \frac{\rho(u)}{x - u} du \quad (2-7)$$

and is continuous except cuts on the support $\cup_i [z_{2i-1}, z_{2i}]$ of the density ρ .

One can show [9, 3, 7] that this resolvent satisfies an algebraic equation:

$$\mathcal{E}(x, W(x)) = 0 \quad (2-8)$$

where $\mathcal{E}(x, y)$ is a polynomial in its variables. Once the so-called spectral curve $\mathcal{E}(x, y) = 0$ is known, one thus obtain $W(x)$ by solving a polynomial equation. Nevertheless, generically there exists as many solutions to this equation as the degree of this polynomial in y . But, for some particular values of x , this equation has double roots in y . These particular points z_i are usually called branch points since they correspond to the points where two different solution of the equation meet.

It is easily seen that these branch points correspond to the z_i fixing the endpoints of the different component of the support of the density ρ . Finding the position of these branch points allows thus to know the support of the eigenvalues.

When the size of the matrices tends to infinity, the eigenvalues fill this support continuously and one is interested in the behavior of the level density inside these support and on their boundaries. This is why we now study this spectral curve in more details.

2.2 Algebraic curve

Following [7], one can compute the spectral curve associated to the considered model. For this model and arbitrary values of the a_i 's: the algebraic equation takes the form:

$$\mathcal{E}(x, y) := (x - y) \prod_{i=1}^k (y - a_i) + P(y) = 0 \quad (2-9)$$

where $P(y)$ is a polynomial of degree $k - 1$ whose coefficients must be fixed by the behavior of the function y at infinity.

Indeed, this algebraic equation can be seen as the realization of a compact Riemann surface Σ embedded in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with two meromorphic functions x and y such that

$$\forall p \in \Sigma, \mathcal{E}(x(p), y(p)) = 0. \quad (2-10)$$

One can easily see that this is a genus 0 curve composed of $k + 1$ -sheets in y . It means that for a given value of $x(p) \in \mathbb{CP}^1$, there exist $k + 1$ points on Σ which are pre-images of x . We note them p^0, p^1, \dots, p^k :

$$\forall i = 0, \dots, k, x(p^i) = x(p). \quad (2-11)$$

To agree with the notations of [3, 4, 5], one can define the value of the function y on the different sheets as $k + 1$ different functions on the complex plane of the image $x(p)$:

Definition 2.1 *Let the functions $\xi_i : \mathbb{C} \rightarrow \mathbb{C}$ be defined by*

$$\xi_i(x(p)) := y(p^i). \quad (2-12)$$

The different sheets can be discriminated by the behavior of $y(p)$ when $x(p) \rightarrow \infty$. From the definition, one can label the different sheets in such a way that:

$$\begin{aligned} y(p^0) &\sim_{x(p) \rightarrow \infty} x(p) + \frac{1}{x(p)} + O\left(\frac{1}{x(p)^2}\right) \\ \forall i \neq 0, \quad y(p^i) &\sim_{x(p) \rightarrow \infty} a_i + \frac{n_i}{N} \frac{1}{x(p)} + O\left(\frac{1}{x(p)^2}\right), \end{aligned} \quad (2-13)$$

i.e. the ξ -functions have the following asymptotics:

$$\begin{aligned} \xi_0(x) &\sim_{x \rightarrow \infty} x + \frac{1}{x} + O\left(\frac{1}{x^2}\right) \\ \forall i \neq 0, \quad \xi_i(x) &\sim_{x \rightarrow \infty} a_i + \frac{n_i}{N} \frac{1}{x} + O\left(\frac{1}{x^2}\right). \end{aligned} \quad (2-14)$$

Inserting these asymptotics inside the algebraic equation Eq. (2-9), one gets the coefficients of the polynomial $P(y)$ and the algebraic curve reads:

$$\mathcal{E}(x, y) = \prod_{i=1}^k (y - a_i) \left(x - y - \sum_{j=1}^k \frac{n_j}{N(y - a_j)} \right).$$

(2-15)

2.3 Kernel and correlation functions

In [6, 8], Daems and Kuijlaars extended classical results of random matrix theory to the case of 1 matrix model in an external field: they explain how the correlation functions can be computed thanks to a unique kernel which can be linked to the solution of an associate Riemann-Hilbert problem.

They showed that, using some multiple orthogonal polynomials, one can express the correlation functions of the model (see [6, 8, 11, 12]). In particular, the joint probability density on the eigenvalues can be written

$$\frac{1}{N!} \det(K_N(x_j, x_k))_{1 \leq j, k \leq N} \quad (2-16)$$

and the m -point correlation functions

$$R_m(x_1, \dots, x_m) = \det(K_N(x_j, x_k))_{1 \leq j, k \leq m} \quad (2-17)$$

where the kernel $K_N(x, y)$ is expressed through some multi-orthogonal polynomials ψ and ϕ by

$$K_N(x, y) = e^{-\frac{1}{2}(x^2 + y^2)} \sum_{k=0}^{N-1} \psi_k(x) \phi_k(y). \quad (2-18)$$

Moreover, in [6, 8], it was shown that these kernels associated to multiple orthogonal polynomials are solution of a general Riemann Hilbert problem. Let us specify here the problem whose solution is the kernel giving correlation functions of the hermitian matrix model introduced above with an external matrix A whose eigenvalues are a_i , $i = 1, \dots, k$.

Theorem 2.1 (*Daems-Kuijlaars*)

The correlation functions are given by

$$R_m(x_1, \dots, x_m) = \det(K_N(x_j, x_k))_{1 \leq j, k \leq m} \quad (2-19)$$

where the kernel $K_N(x, y)$ reads:

$$K_N(x, y) = \frac{1}{2i\pi(x-y)} \Omega_1(y) Y_+^{-1}(y) Y_+(x) \Omega_2^t(x) \quad (2-20)$$

where we define the vectors of size $k+1$:

$$\Omega_1(x) = [0 \ e^{N\frac{x^2}{4}} \omega_1(x) \ \dots \ e^{N\frac{x^2}{4}} \omega_k(x)] \quad (2-21)$$

with $\omega_i := e^{-N(\frac{x^2}{2} - a_i x)}$ and

$$\Omega_2(x) = [1 \ 0 \ \dots \ 0], \quad (2-22)$$

and $Y(x)$ is the unique solution of the following Riemann Hilbert problem:

- $Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{(k+1) \times (k+1)}$ is analytic;
- for $x \in \mathbb{R}$, one has the jumps

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega_1(x) & \omega_2(x) & \dots & \omega_k(x) \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad (2-23)$$

where $Y_+(x)$ and $Y_-(x)$ denote respectively the limit of $Y(z)$ when $z \rightarrow x$ from the upper and lower half planes;

- when $x \rightarrow \infty$, one has the asymptotic behavior:

$$Y(x) = \left(\mathbf{I} + O\left(\frac{1}{x}\right) \right) \begin{pmatrix} z^N & 0 & 0 & \dots & 0 \\ 0 & z^{-n_1} & 0 & \dots & 0 \\ 0 & 0 & z^{-n_2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & z^{-n_k} \end{pmatrix} \quad (2-24)$$

where \mathbf{I} denotes the identity matrix of size $(k+1) \times (k+1)$.

2.4 Non-intersecting Brownian motions

One of the main motivations for this work is the study of a physical system composed of many N Brownian motions as N gets large. Let us consider N Brownian motions $x_i(t)$, $i = 1, \dots, N$, on the real axis starting from the origin at time $t = 0$:

$$\forall i = 1, \dots, N, \quad x_i(0) = 0. \quad (2-25)$$

Let us impose that they cannot intersect and that their final positions at time t is given by a set of k ordered distinct real points a_i :

$$(x_1(1), x_2(1), x_3(1), \dots, x_N(1)) = \left(\overbrace{a_1, \dots, a_1}^{n_1}, \overbrace{a_2, \dots, a_2}^{n_2}, \dots, \overbrace{a_k, \dots, a_k}^{n_k} \right) \quad (2-26)$$

for any integers $\{n_i\}_{i=1}^k$ such that

$$\sum_{i=1}^k n_i = N. \quad (2-27)$$

The physical quantities describing this statistical system at a given time $0 < t < 1$ are the correlation functions:

$$R_m(X_1, X_2, \dots, X_M; t) := \frac{N!}{(N-m)!} \int_{\mathbb{R}^{N-m}} dP(x_1(t), x_2(t), \dots, x_N(t)) \prod_{i=1}^m \delta(x_i - X_i) \quad (2-28)$$

where $P(x_1(t), x_2(t), \dots, x_N(t))$ is the probability measure on the Brownian motions at time t .

This problem does not look easy to solve at first sight. Fortunately, there exists an underlying integrable structure and it was proved that these correlation functions coincide with the correlation functions of a hermitian gaussian matrix model submitted to an external matrix

$$A := A(t) := \text{diag}(\overbrace{a_1(t), \dots, a_1(t)}^{n_1}, \overbrace{a_2(t), \dots, a_2(t)}^{n_2}, \dots, \overbrace{a_k(t), \dots, a_k(t)}^{n_k}) \quad (2-29)$$

where the eigenvalues a_i are just rescaled versions of the ending points:

$$a_i(t) := a_i \sqrt{\frac{t}{1-t}}. \quad (2-30)$$

In order to compute these correlation functions at time t , one is then left with the study of the corresponding random matrix $M(t)$ whose statistic depends on t only through the external matrix eigenvalues $a_i(t)$. Remark also that the time evolution of these external eigenvalues is very simple.

Let us now interpret the ingredients of the matrix model's study in this statistical physics setup to get some intuition. First of all, one sees that the eigenvalues of the random matrix $x_i(t)$ are identified with the position of the Brownian motions at time t . The cuts of the spectral curve $[z_{2i-1}(t), z_{2i}(t)]_{i=1}^l$ should then coincide with the support of the Brownian motions at time t . One can then study the evolution of these support by studying the time evolution of their edges, i.e. the time evolution of the branch points.

Consider the equation of the algebraic curve Eq. (2-15). The inversion formula giving x in terms of y reads³

$$\boxed{x = y + \sum_{i=1}^k \frac{n_i}{N(y - a_i(t))}} \quad (2-32)$$

and has, generically, $2k$ branch points given by the roots of the equation:

$$1 = \sum_{i=1}^k \frac{n_j}{N(y - a_j(t))^2}. \quad (2-33)$$

Let us call $\{z_i(t)\}_{i=1}^{2k}$ these branch points ordered following their real part. Since the a_j 's are real, the roots of Eq. (2-33) are either real or come in pairs of complex conjugated points.

Let us now have a look at the two extremal cases corresponding to the boundaries. For $t \rightarrow 1$, the eigenvalues diverge $a_i(t) \rightarrow \infty$. Thus, the equation for the branch points Eq. (2-33) as solutions:

$$y \sim a_i(1) \sim \infty, \quad (2-34)$$

i.e. one has $2k$ real branch points of the type $z_{2i-1} < a_i < z_{2i}$ with z_{2i-1} and z_{2i} very close to a_i . This confirms the physical intuition of the behavior of the Brownian motions: they merge in k groups close to the end points a_i when $t \rightarrow 1$. On the contrary, as $t \rightarrow 0$, all the $a_i(t)$ merge to 0 and all the branch points merge also to 0: only two of them are real, the other $2k - 2$ come in pairs of complex conjugated points.

Let us now come to the generic configuration. Starting from $t = 1$, as the time decreases the real branch points z_{2i-1} and z_{2i} move away from a_i . When z_{2i} and z_{2i+1} meet, one reaches a critical point and the corresponding configuration is not studied

³Remark that this gives a natural rational parametrization of this curve:

$$\begin{cases} y = z \\ x = z + \sum_{i=1}^k \frac{n_i}{N(z - a_i(t))} \end{cases} \quad (2-31)$$

in this paper but will be addressed elsewhere. After crossing this phase transition, these two branch points move away from the real axis becoming complex conjugated. Then, at any intermediate time, a generic configuration consists in $2l$ real branch points (z_{2i-2}, z_{2i}) , $i = 1, \dots, l$, where the number of real cuts l decreases with time from k to 1^4 . The other branch points are complex conjugated numbers $(w_i^{(h)}, \overline{w}_i^{(h)})$ whose real part belong to the segments $[z_{2i-2}, z_{2i}]$.

In the following sections, we study the Riemann surface associated to these different configuration: in section 3, we consider the case where we have only real branch points whereas in section 4, we study the generic case with an arbitrary number of real eigenvalues.

3 Large time case

In this section, we assume that all the branch points are real. This configuration is encountered for large times in the brownian motion setup: it corresponds to having all the particles in the neighborhood of their end points. Then one can simply mimic the computations of Bleher and Kuijlaars [3] and study the algebraic curve.

3.1 Study of the spectral curve

Before solving the Riemann-Hilbert problem, one needs to study the associated spectral curve more carefully. Remember that, in the large time case we consider in this paragraph, the spectral curve has $2k$ real branch points $\{z_i\}_{i=1}^{2k}$.

We can show the following lemmas concerning the sheet structure of the spectral curve built out of the y function. First of all, we can build analytical continuations of the ξ_i as follows:

Lemma 3.1 *The functions ξ_i for $i \neq 0$ (resp. ξ_0) are analytic in $\mathbb{C} \setminus [z_{2i-1}, z_{2i}]$ (resp. $\mathbb{C} \setminus \bigcup_{j=1}^k [z_{2j-1}, z_{2j}]$).*

proof:

It is easily proved by starting from the asymptotics Eq. (2-14) and analytically continuing these functions using the rational parametrization of this genus 0 curve.

□

These functions satisfy the following jump conditions:

⁴The number l corresponds to the number of disjoint segments filled by the Brownian motions at a given time t . The proof of this property is not technically involved but rather long, that is why it was decided not to include it in the present paper since the interested reader can easily derive it.

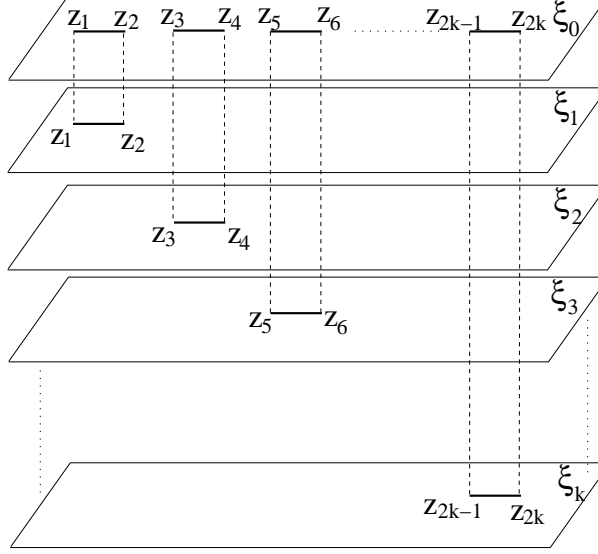


Figure 1: Sheeted structure of the spectral curve in the large time case. The sheet 0 is linked to the other k sheets by non intersecting segments.

Lemma 3.2 For any $i = 1, \dots, k$ and $x \in [z_{2i-1}, z_{2i}]$

$$\xi_{0+}(x) = \overline{\xi_{0-}(x)} = \xi_{i-}(x) = \overline{\xi_{i+}(x)} \quad (3-1)$$

and for $x \in \bigcup_{i=1}^k]z_{2i-1}, z_{2i}[$

$$\text{Im}\xi_{0+}(x) > 0 \quad (3-2)$$

proof:

It also follows directly from the definitions. \square

It means that the spectral curve is composed of $k + 1$ sheets linked by k real cuts as follows: the sheet i is linked to the sheet 0 by the cut $[z_{2i-1}, z_{2i}]$ (see fig.1).

We can then define the "eigenvalue density" corresponding to the density of brownian particles at a given point x of the real axis:

Definition 3.1 Let us define, for any $x \in \bigcup_{j=1}^k [z_{2j-1}, z_{2j}]$,

$$\rho(x) := \frac{1}{\pi} \text{Im}\xi_{0+}(x). \quad (3-3)$$

One can now study its properties giving the physically expected result:

Lemma 3.3 For $x \in \bigcup_{i=1}^k]z_{2i-1}, z_{2i}[$, $\rho(x) > 0$ and

$$\int_{z_{2i-1}}^{z_{2i}} \rho(x) dx = \frac{n_i}{N}. \quad (3-4)$$

On the edges of the spectrum, there exists $\rho_i > 0$ such that

$$\rho(x) = \frac{\rho_i}{\pi} |x - z_i| (1 + O(z - z_i)) \text{ as } x \rightarrow z_i. \quad (3-5)$$

proof:

From the preceding lemma, one directly get that

$$\forall x \in \bigcup_{i=1}^k]z_{2i-1}, z_{2i}[, \rho(x) > 0. \quad (3-6)$$

Consider now any integer $i \in [1, k]$ and $x \in]z_{2i-1}, z_{2i}[$, one has

$$\rho(x) = \frac{1}{\pi} \text{Im} \xi_{0+}(x) = \frac{1}{2i\pi} (\xi_{0+}(x) - \overline{\xi_{0+}(x)}) = \frac{1}{2i\pi} (\xi_{i-}(x) - \xi_{i+}(x)) \quad (3-7)$$

and thus, one can compute the integral

$$\int_{z_{2i-1}}^{z_{2i}} \rho(x) dx = \frac{1}{2i\pi} \int_{z_{2i-1}}^{z_{2i}} (\xi_{i-}(x) - \xi_{i+}(x)) dx = \frac{1}{2i\pi} \oint_{\Gamma_i} \xi_i(x) dx \quad (3-8)$$

where the integration contour Γ_i encircles the cut $[z_{2i-1}, z_{2i}]$. Since ξ_i has no other poles except infinity, one can move it and use the asymptotics:

$$\int_{z_{2i-1}}^{z_{2i}} \rho(x) dx = - \text{Res}_{x \rightarrow \infty} \xi_i(x) dx = \frac{n_i}{N}. \quad (3-9)$$

The last part of the lemma follows from the usual behavior on the edge of the spectrum (see for example [3]). \square

Remark 3.1 All the results of this lemma simply follows on the behavior of the 1-form $y dx$ in the neighborhood of a cut linking two simple branch points. Thus all these *local* result could be obtained by the former study of simpler models where one has only two sheets for example.

We now define the integrals of the ξ_i functions as:

Definition 3.2 We define the integrals λ_k by

$$\lambda_k(z) := \int^z \xi_k(x) dx \quad (3-10)$$

where the integration constants are constrained by:

$$\lambda_0(z_{2k}) = \lambda_k(z_{2k}) = 0 \quad (3-11)$$

and

$$\forall i = 1, \dots, k-1, \lambda_i(z_{2i}) = \lambda_{0+}(z_{2i}). \quad (3-12)$$

Lemma 3.4 *With this definition, the jumps and asymptotics of these λ -functions are given by:*

- When $z \rightarrow \infty$:

$$\lambda_0(z) = \frac{z^2}{2} - \ln z + l_0 + O\left(\frac{1}{z^2}\right) \quad (3-13)$$

and, for $i \neq 0$:

$$\lambda_i(z) = a_i z + \frac{n_i}{N} \ln z + l_i + O\left(\frac{1}{z^2}\right) \quad (3-14)$$

where l_i 's are some constants chosen to satisfy the constraints Eq. (3-11) and Eq. (3-12).

- They follow the jumps:

$$\begin{aligned} \lambda_{0+}(x) - \lambda_{0-}(x) &= -2i\pi \sum_{j=i+1}^k \frac{n_j}{N} & x \in [z_{2i}, z_{2i+1}] \\ \lambda_{0+}(x) - \lambda_{0-}(x) &= -2i\pi & x \in (-\infty, z_1] \\ \forall i \in [1, k], \lambda_{i+}(x) - \lambda_{i-}(x) &= 2i\pi \frac{n_i}{N} & x \in (-\infty, z_{2i-1}] \\ \lambda_{0+}(x) - \lambda_{k-}(x) &= 0, \lambda_{0-}(x) - \lambda_{k+}(x) = 0 & x \in [z_{2k-1}, z_{2k}] \\ \forall i \in [1, k-1], \lambda_{0+}(x) - \lambda_{i-}(x) &= 0, \lambda_{0-}(x) - \lambda_{i+}(x) = 2i\pi \sum_{j=i+1}^k \frac{n_j}{N} & x \in [z_{2i-1}, z_{2i}] \end{aligned} \quad (3-15)$$

proof:

The first part of this lemma is just the rewriting of the integration of the asymptotics of ydx on the different sheets.

The second part is obtained by moving the integration contours so that one is left by the computation of a residue at infinity on the different sheets. \square

Let us now state a fundamental theorem for the following

Theorem 3.1 *For any $i = 1, \dots, k$, on $\mathbb{R} \setminus [z_{2i-1}, z_{2i}]$, one has $\operatorname{Re} \lambda_{i+} < \operatorname{Re} \lambda_{0-}$.*

For any $i = 1, \dots, k$, there exists a neighborhood U_i of $]z_{2i-1}, z_{2i}[$ such that

$$\forall j \neq i, \operatorname{Re} \lambda_j(z) < \operatorname{Re} \lambda_0(z) < \operatorname{Re} \lambda_i(z) \quad (3-16)$$

for any $z \in U_i -]z_{2i-1}, z_{2i}[$.

proof:

Once again, one can follow [3] for the first part of the theorem: for $x > z_{2i}$, $\xi_0(x) > \xi_i(x)$, and $\lambda_0(x) > \lambda_i(x)$ because $\lambda_i(z_{2i}) = \lambda_0(z_{2i})$ and $\lambda'_i = \xi_i$; for $x < z_{2i-1}$, one easily sees that $\operatorname{Re} \xi_0(x) < \xi_i(x)$ which gives the result.

For the second part, one has to study the properties of the λ -functions more carefully. Consider the function $F_i := \lambda_{i+} - \lambda_{0+}$. It is purely imaginary on $]z_{2i-1}, z_{2i}[$ and its derivative $F'_i(x) = -2i\pi\rho(x)$ has negative imaginary part. Thus $\operatorname{Re} \lambda_0(z) < \operatorname{Re} \lambda_i(z)$ in a neighborhood of $]z_{2i-1}, z_{2i}[$. The other inequality follows from the definition and continuity. \square

3.2 Riemann-Hilbert analysis

In this section, we solve the Riemann-Hilbert problem leading to the computation of the kernel. We look for the unique function Y defined as follows:

Definition 3.3 *Let Y be the solution of the following constraints:*

- $Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{(k+1) \times (k+1)}$ is analytic;
- for $x \in \mathbb{R}$, $Y(x)$ has the jumps

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega_1(x) & \omega_2(x) & \dots & \omega_k(x) \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad (3-17)$$

where $\omega_i(x) := e^{-N(\frac{x^2}{2} - a_i x)}$ and $Y_+(x)$ and $Y_-(x)$ denote respectively the limit of $Y(z)$ when $z \rightarrow x$ from the upper and lower half planes;

- when $x \rightarrow \infty$, one has the asymptotic behavior:

$$Y(x) = \left(\mathbf{I} + O\left(\frac{1}{x}\right) \right) \begin{pmatrix} z^N & 0 & 0 & \dots & 0 \\ 0 & z^{-n_1} & 0 & \dots & 0 \\ 0 & 0 & z^{-n_2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & z^{-n_k} \end{pmatrix} \quad (3-18)$$

The kernel can then be written:

$$K_N(x, y) = \frac{1}{2i\pi(x-y)} \Omega_1(y) Y_+^{-1}(y) Y_+(x) \Omega_2^t(x) \quad (3-19)$$

with the notations of Eq. (2-21) and Eq. (2-22). To do so, we transform the problem step by step to turn it to a simple one (see [2] for a review on the subject).

It is important to keep in mind the constraints we impose on the n'_i 's and N :

- We fix a set of k positive rational number called filling fractions:

$$\epsilon_i = \frac{p_i}{q_i} \quad (3-20)$$

where p_i and q_i are coprime and

$$\sum_{i=1}^k \epsilon_i = 1. \quad (3-21)$$

- N is a common multiple of the q_i 's.
- The integers n_i are fixed by the condition:

$$n_i = \epsilon_i N. \quad (3-22)$$

With these constraints, the spectral curve associated to this RH problem does not change as N grows.

3.2.1 First transformation

Using the notations of section 3.1, we can define

$$T(x) := \text{diag} \left(e^{-Nl_0}, e^{-Nl_1}, \dots, e^{-Nl_k} \right) Y(x) \text{diag} \left(e^{N(\lambda_0 - \frac{x^2}{2})}, e^{N(\lambda_1 - \frac{n_1}{N}x)}, e^{N(\lambda_2 - \frac{n_2}{N}x)}, \dots, e^{N(\lambda_k - \frac{n_k}{N}x)} \right) \quad (3-23)$$

for $x \in \mathbb{C} \setminus \mathbb{R}$.

Then one can check that it is solution of the Riemann-Hilbert problem:

- T is analytic on $\mathbb{C} \setminus \mathbb{R}$;
- for $x \in \mathbb{R}$, one has the jumps

$$T_+(x) = T_-(x) j_T(x) \quad (3-24)$$

where

$$j_T(x) = \begin{pmatrix} e^{N(\lambda_{0+}(x) - \lambda_{0-}(x))} & e^{N(\lambda_{1+}(x) - \lambda_{0-}(x))} & e^{N(\lambda_{2+}(x) - \lambda_{0-}(x))} & \dots & e^{N(\lambda_{k+}(x) - \lambda_{0-}(x))} \\ 0 & e^{N(\lambda_{1+}(x) - \lambda_{1-}(x))} & 0 & \dots & 0 \\ 0 & 0 & e^{N(\lambda_{2+}(x) - \lambda_{2-}(x))} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & e^{N(\lambda_{k+}(x) - \lambda_{k-}(x))} \end{pmatrix}. \quad (3-25)$$

- when $x \rightarrow \infty$, one has the asymptotic behavior:

$$T(x) = \left(\mathbf{I} + O\left(\frac{1}{x}\right) \right). \quad (3-26)$$

Note that one can write the jump matrix $j_T(x)$ depending on the cut on which x lies:

For $x \in [z_{2i-1}, z_{2i}]$, j_T takes the form

$$\begin{pmatrix} e^{N(\lambda_0(x) - \lambda_i(x)) +} & e^{N(\lambda_{1+}(x) - \lambda_{0-}(x))} & \dots & e^{N(\lambda_{i-1+}(x) - \lambda_{0-}(x))} & 1 & e^{N(\lambda_{i-1+}(x) - \lambda_{0-}(x))} & \dots & e^{N(\lambda_{k+}(x) - \lambda_{0-}(x))} \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & e^{N(\lambda_0(x) - \lambda_i(x)) -} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (3-27)$$

For $x \in \mathbb{R} \setminus \bigcup_i [z_{2i-1}, z_{2i}]$, j_T reduces to

$$j_T = \begin{pmatrix} 1 & e^{N(\lambda_{1+}(x) - \lambda_{0-}(x))} & e^{N(\lambda_{2+}(x) - \lambda_{0-}(x))} & \dots & e^{N(\lambda_{k+}(x) - \lambda_{0-}(x))} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (3-28)$$

The kernel can then be written:

$$K_N(x, y) = \frac{e^{N\left(\frac{x^2}{4} - \frac{y^2}{4}\right)}}{2i\pi(x - y)} \Omega_{1,T}(y) T_+^{-1}(y) T_+(x) \Omega_{2,T}^t(x) \quad (3-29)$$

where

$$\Omega_{1,T}(x) = [0 \ e^{N\lambda_{1+}(x)} \ \dots \ e^{N\lambda_{k+}(x)}] \quad (3-30)$$

and

$$\Omega_{2,T}(x) = [e^{-N\lambda_{0+}(x)} \ 0 \ \dots \ 0]. \quad (3-31)$$

3.2.2 Second transformation and opening of lenses

Remark that one can further factorize the matrix $j_T(x)$ for $x \in [z_{2i-1}, z_{2i}]$:

$$j_T(x) = \tilde{j}_{i,T}(x) j_S(x) \hat{j}_{i,T}(x) \quad (3-32)$$

where, for $x \in [z_{2i-1}, z_{2i}]$, one has defined the $k+1 \times k+1$ matrices:

$$j_S(x) = [(j_S)_{\alpha\beta}]_{\alpha,\beta=0}^k := [(1 - \delta_{\alpha i} - \delta_{\alpha 0})\delta_{\alpha\beta} + \delta_{\alpha 0}\delta_{\beta i} - \delta_{\alpha i}\delta_{\beta 0}]_{\alpha,\beta=0}^k, \quad (3-33)$$

$$\tilde{j}_{i,T}(x) := \left[\delta_{\alpha\beta} + \delta_{\alpha i}(\delta_{\beta 0} e^{N(\lambda_0 - \lambda_i)_-} - \sum_{j \neq 0, i} \delta_{\beta j} e^{N(\lambda_j - \lambda_i)_-}) \right]_{\alpha,\beta=0}^k \quad (3-34)$$

and

$$\hat{j}_{i,T}(x) := \left[\delta_{\alpha\beta} + \delta_{\alpha i} \left(\sum_{j \neq i} \delta_{\beta j} e^{N(\lambda_j - \lambda_i)_+} \right) \right]_{\alpha,\beta=0}^k. \quad (3-35)$$

This allows us to define the second transformation. Consider a set of k lenses with vertices z_{2i-1} and z_{2i} (see fig.2) contained in the neighborhood U_i defined in lemma 3.1 and define

$$S(z) := \begin{cases} T(z) \hat{J}_i(x) & \text{in the upper lens region} \\ T(z) \tilde{J}_i(x) & \text{in the lower lens region} \end{cases} \quad (3-36)$$

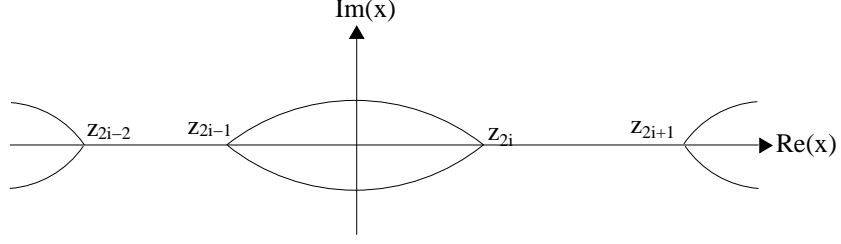


Figure 2: Lenses used in the first transformation: the lense i has vertices z_{2i-1} and z_{2i} .

with

$$\tilde{J}_i(x) := \left[\delta_{\alpha\beta} + \delta_{\alpha i}(\delta_{\beta 0} e^{N(\lambda_0 - \lambda_i)} - \sum_{j \neq 0, i} \delta_{\beta j} e^{N(\lambda_j - \lambda_i)}) \right]_{\alpha, \beta=0}^k \quad (3-37)$$

and

$$\hat{J}_i(x) := \left[\delta_{\alpha\beta} - \delta_{\alpha i} \left(\sum_{j \neq i} \delta_{\beta j} e^{N(\lambda_j - \lambda_i)} \right) \right]_{\alpha, \beta=0}^k. \quad (3-38)$$

for z in the lens i , and

$$S(z) := T(z) \quad (3-39)$$

for z outside the lenses.

We now define the jump matrix $j_S(x)$ on all the discontinuity contours of $S(x)$:

- On $[z_{2i-1}, z_{2i}]$, for $i = 1, \dots, k$:

$$j_S(x) = [(1 - \delta_{\alpha i} - \delta_{\alpha 0})\delta_{\alpha\beta} + \delta_{\alpha 0}\delta_{\beta i} - \delta_{\alpha i}\delta_{\beta 0}]_{\alpha, \beta=0}^k, \quad (3-40)$$

- On the upper boundary of the lens i :

$$j_S(x) = \left[\delta_{\alpha\beta} - \delta_{\alpha i} \left(\sum_{j \neq i} \delta_{\beta j} e^{N(\lambda_j - \lambda_i)} \right) \right]_{\alpha, \beta=0}^k, \quad (3-41)$$

- On the lower boundary of the lens i :

$$j_S(x) = \left[\delta_{\alpha\beta} + \delta_{\alpha i}(\delta_{\beta 0} e^{N(\lambda_0 - \lambda_i)} - \sum_{j \neq 0, i} \delta_{\beta j} e^{N(\lambda_j - \lambda_i)}) \right]_{\alpha, \beta=0}^k, \quad (3-42)$$

- On $\mathbb{R} \setminus \bigcup_i [z_{2i-1}, z_{2i}]$:

$$j_S(x) = j_T(x). \quad (3-43)$$

Using these definition, one sees that $S(x)$ is solution of the following RH problem:

- $S(x)$ is analytic on $\mathbb{C} \setminus (\mathbb{R} \cup \Gamma)$ where Γ denotes the boundaries of the lenses;
- $S(x)$ has jumps:

$$\forall x \in \mathbb{R} \cup \Gamma, \quad S_+(x) = S_-(x)j_S(x); \quad (3-44)$$

- when $x \rightarrow \infty$:

$$S(x) = \mathbf{I} + O\left(\frac{1}{x}\right) \quad (3-45)$$

and the kernel can be expressed in terms of this matrix S by the following expression:
for $(x, y) \in (z_{2i-1}, z_{2i})$

$$K_N(x, y) = \frac{e^{N[h(y)-h(x)]}}{2i\pi(x-y)} \Omega_{1,S,i}(y) S_+^{-1}(y) S_+(x) \Omega_{2,S,i}^t(x) \quad (3-46)$$

where

$$h(x) = -\frac{x^2}{4} + \operatorname{Re}[\lambda_{0+}(x)], \quad (3-47)$$

$$\Omega_{1,S,i}(x) = [-e^{Ni\operatorname{Im}[\lambda_{0+}(x)]} \overbrace{0 \dots 0}^{i-1} e^{-Ni\operatorname{Im}[\lambda_{0+}(x)]} \overbrace{0 \dots 0}^{k-i}] \quad (3-48)$$

and

$$\Omega_{2,S,i}(x) = [e^{-Ni\operatorname{Im}[\lambda_{0+}(x)]} \overbrace{0 \dots 0}^{i-1} e^{Ni\operatorname{Im}[\lambda_{0+}(x)]} \overbrace{0 \dots 0}^{k-i}]. \quad (3-49)$$

One can see that, as $N \rightarrow \infty$, this RH problem reduces to a model RH of the same kind of that studied in [3, 4, 5] thanks to the properties of the λ -functions in section 3.1. Let us solve it in the same way.

3.2.3 Model RH problem.

This means that we want to solve the following RH problem:

- $M(x) : \mathbb{C} \setminus \bigcup_{i=1}^k [z_{2i-1}, z_{2i}] \rightarrow \mathbb{C}^{k+1 \times k+1}$;
- $M(x)$ is analytic in $\mathbb{C} \setminus \bigcup_{i=1}^k [z_{2i-1}, z_{2i}]$;
- M has jumps on the cuts $[z_{2i-1}, z_{2i}]$ given by:

$$\forall x \in \bigcup_{i=1}^k [z_{2i-1}, z_{2i}], \quad M_+(x) = M_-(x)j_S(x); \quad (3-50)$$

- as $x \rightarrow \infty$:

$$M(x) = \mathbf{I} + O\left(\frac{1}{x}\right). \quad (3-51)$$

Let us lift this problem to the associated spectral curve by cutting the complex plane along the different y -sheets: one defines:

$$\Omega_i = \xi_i(\mathbb{C} \setminus [z_{2i-1}, z_{2i}]) \quad (3-52)$$

for $i = 1, \dots, k$ and

$$\Omega_0 = \xi_0(\mathbb{C} \setminus \bigcup_{i=1}^k [z_{2i-1}, z_{2i}]). \quad (3-53)$$

These images of the different y -sheets give a partition of the complex plane by definition and we can define the boundaries $\Gamma_i = \Gamma_i^+ \cup \Gamma_i^-$ separating Ω_0 and Ω_i for any $i \neq 0$ as follows:

$$\Gamma_i^+ = \Gamma_i \cap \{\text{Im} z \geq 0\} = \xi_{0+}([z_{2i-1}, z_{2i}]) = \xi_{i-}([z_{2i-1}, z_{2i}]) \quad (3-54)$$

and

$$\Gamma_i^- = \Gamma_i \cap \{\text{Im} z \leq 0\} = \xi_{0-}([z_{2i-1}, z_{2i}]) = \xi_{i+}([z_{2i-1}, z_{2i}]). \quad (3-55)$$

We are looking for a solution under the form of a Lax matrix

$$M(x) = \begin{pmatrix} \phi_0(\xi_0(x)) & \phi_0(\xi_1(x)) & \dots & \phi_0(\xi_k(x)) \\ \phi_1(\xi_0(x)) & \phi_1(\xi_1(x)) & \dots & \phi_1(\xi_k(x)) \\ \dots & \dots & \dots & \dots \\ \phi_k(\xi_0(x)) & \phi_k(\xi_1(x)) & \dots & \phi_k(\xi_k(x)) \end{pmatrix} \quad (3-56)$$

where ϕ is a Baker-Akhiezer type function analytic on $\mathbb{C} \setminus \bigcup_i \Gamma_i$ which must satisfy the jump conditions:

$$\forall x \in \bigcup_j \Gamma_j^+, \phi_{i+}(\xi) = \phi_{i-}(\xi) \quad (3-57)$$

and

$$\forall x \in \bigcup_j \Gamma_j^-, \phi_{i+}(\xi) = -\phi_{i-}(\xi). \quad (3-58)$$

These functions are also constrained by their asymptotic behavior:

$$\forall i, j = 0, \dots, k, \phi_i(a_j) = \phi_i(\xi_j(\infty)) := \delta_{ij} \quad (3-59)$$

where one conventionally notes $a_0 := \infty$.

One also notes

$$\Gamma_i \cap \mathbb{R} = \{p_{2i-1}, p_{2i}\} = \{\xi_i(z_{2i-1}), \xi_i(z_{2i})\}. \quad (3-60)$$

One has the solution:

$$\phi_0(\xi) = \frac{\prod_{i=1}^k (\xi - a_i)}{\sqrt{\prod_{i=1}^k (\xi - p_{2i-1})(\xi - p_{2i})}} \quad (3-61)$$

and

$$\phi_i(x) = c_i \frac{\prod_{j \neq i} (\xi - a_j)}{\sqrt{\prod_{i=1}^k (\xi - p_{2i-1})(\xi - p_{2i})}} \quad (3-62)$$

for $i = 1, \dots, k$, where the c_i 's are constants which can be fixed by the asymptotic behavior of ϕ_i 's and the square roots have branch cuts along $\bigcup \Gamma_i$.

Indeed, the algebraic equation defining the branch points reads:

$$\prod_{i=1}^k (\xi - p_{2i-1})(\xi - p_{2i}) = \prod_{i=1}^k (\xi - a_i)^2 - \sum_{i=1}^k \frac{n_i}{N} \prod_{j \neq i} (\xi - a_j)^2 \quad (3-63)$$

which gives

$$\phi_i(a_i) = c_i \sqrt{-\frac{N}{n_i}} \quad (3-64)$$

i.e.

$$c_i = -i \sqrt{\frac{n_i}{N}}. \quad (3-65)$$

3.2.4 Parametrix at the edge points

Now the solution of the RH problem can be build away from the branch points thanks to the preceding section, let us have a closer look at the branch points z_j . Since the behavior near the branch points is the typical square root and one can use the usual method (see for example [3]).

The problem is to find a parametrix in the neighborhood of the branch points, i.e. a solution to the RH problem:

- $P(z)$ is analytic in $\bigcup_{i=1}^{2k} D(z_i, r) \setminus (\mathbb{R} \cup \Gamma)$ where $D(z_i, r)$ is a disc centered at z_i with small radius r and Γ is the set lenses defined used for the second transformation;
- P has jumps:

$$\forall z \in (\mathbb{R} \cup \Gamma) \cap \left(\bigcup_{i=1}^{2k} D(z_i, r) \right), \quad P_+(z) = P_-(z) j_S(z); \quad (3-66)$$

- When $n \rightarrow \infty$, the boundary conditions for P are:

$$\forall i = 1, \dots, 2k, \quad \forall z \in \partial D(z_i, r), \quad P(z) = \left(\mathbf{I} + O\left(\frac{1}{N}\right) \right) M(z) \quad \text{uniformly.} \quad (3-67)$$

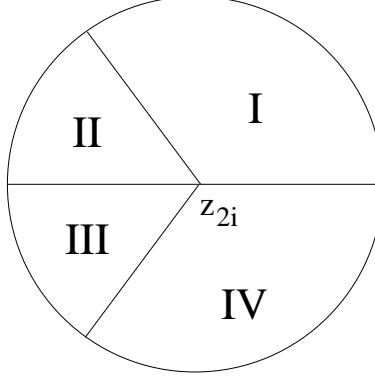


Figure 3: Decomposition of the disc $D(z_{2i}, r)$ into four sectors.

Let us now consider a particular branch point z_{2i} ⁵. When $z \rightarrow z_{2i}$, lemma 3.3 states that:

$$\begin{aligned}\lambda_0(z) &= p_{2i}(z - z_{2i}) + \frac{2\rho_{2i}}{3}(z - z_{2i})^{\frac{3}{2}} + O((z - z_{2i})^2) \\ \lambda_i(z) &= p_{2i}(z - z_{2i}) - \frac{2\rho_{2i}}{3}(z - z_{2i})^{\frac{3}{2}} + O((z - z_{2i})^2)\end{aligned}\tag{3-68}$$

i.e.

$$\lambda_0(z) - \lambda_i(z) = \frac{4\rho_{2i}}{3}(z - z_{2i})^{\frac{3}{2}} + O((z - z_{2i})^{\frac{5}{2}})\tag{3-69}$$

which means that the function $\beta_i(z) = [\frac{3}{4}(\lambda_0(z) - \lambda_i(z))]^{\frac{2}{3}}$ is analytic in a neighborhood of z_{2i} with a strictly positive derivative $\beta'_i(z_{2i}) = \rho_{2i}^{\frac{2}{3}} > 0$.

Thus β_i is a conformal map from $D(z_{2i}, r)$ to a neighborhood of the origin for r small enough.

Let us now cut the image of $D(z_{2i}, r)$ by a special choice of Γ representing the cuts of the lens based in z_{2i} :

$$\beta_i(\Gamma \cap D(z_{2i}, r)) = \left\{ z \mid \arg(z) = \pm \frac{2\pi}{3} \right\}.\tag{3-70}$$

It gives a partition of the disc $D(z_{2i}, r)$ into four regions labeled I, II, III and IV (see figure 3) following:

- $0 < \arg(\beta(z)) < \frac{2\pi}{3}$ in region I;
- $\frac{2\pi}{3} < \arg(\beta(z)) < \pi$ in region II;
- $-\pi < \arg(\beta(z)) < -\frac{2\pi}{3}$ in region III;
- $-\frac{2\pi}{3} < \arg(\beta(z)) < 0$ in region IV.

⁵We could do exactly the same for a branch point z_{2i-1} with the same function β_i .

We can now transform the RH for P to a model one by the transformation:

$$\tilde{P}(z) := \begin{cases} P(z)T_{\tilde{P}P}(z) & \text{in regions I and IV.} \\ P(z) & \text{in regions II and III.} \end{cases} \quad (3-71)$$

where one has defined:

$$T_{\tilde{P}P}(z) = \left[\delta_{\alpha\beta} - \delta_{\alpha i} \sum_{j \neq 0, i} \delta_{\beta j} e^{N(\lambda_j - \lambda_i)} \right]_{\alpha, \beta=0}^k. \quad (3-72)$$

Thus, \tilde{P} is solution of the same RH problem as P except that its jumps are now given by

$$\forall z \in (\mathbb{R} \cup \Gamma) \cap \left(\bigcup_{i=1}^{2k} D(z_i, r) \right), \quad P_+(z) = P_-(z)j_{\tilde{P}}(z) \quad (3-73)$$

where $j_{\tilde{P}}(z)$ is given by:

- for z on $[z_{2i} - r, z_{2i}]$:

$$j_{\tilde{P}} = [(1 - \delta_{\alpha 0} - \delta_{\alpha i})\delta_{\alpha\beta} + \delta_{\alpha 0}\delta_{\beta i} - \delta_{\alpha i}\delta_{\beta 0}]_{\alpha, \beta=0}^k; \quad (3-74)$$

- for z on the upper and lower sides of the lens in $D(z_{2i}, r)$:

$$j_{\tilde{P}} = [\delta_{\alpha\beta} + \delta_{\alpha i}\delta_{\beta 0}e^{N(\lambda_0 - \lambda_i)}]_{\alpha, \beta=0}^k; \quad (3-75)$$

- for z on $]z_{2i}, z_{2i} + r]$:

$$j_{\tilde{P}} = [\delta_{\alpha\beta} + \delta_{\alpha 0}\delta_{\beta i}e^{N(\lambda_0 - \lambda_i)}]_{\alpha, \beta=0}^k. \quad (3-76)$$

One sees that these jumps are equal to the identity except for a 2×2 block corresponding to the $(0, 0)$, $(0, i)$, $(i, 0)$ and $(0, 0)$ coefficients. The RH problem reduces then to a model problem of size 2×2 which can be solved explicitly as follows (see [3] for example):

$$\tilde{P}(z) = E_N(z)\Phi\left(N^{\frac{2}{3}}\beta_i(z)\right)D(z)$$

(3-77)

where E_N ensures the boundary conditions

$$E_N = \sqrt{\pi}M\widetilde{M}\widehat{M}, \quad (3-78)$$

with

$$\widetilde{M} = [(1 - \delta_{\alpha i}(1 + i))\delta_{\alpha\beta} - \delta_{\alpha 0}\delta_{\beta i} - i\delta_{\alpha i}\delta_{\beta 0}]_{\alpha, \beta=0}^k, \quad (3-79)$$

and

$$\widehat{M} = \left[(1 - \delta_{\alpha 0}(1 - n^{\frac{1}{6}}\beta^{\frac{1}{4}}) - \delta_{\alpha i}(1 - n^{-\frac{1}{6}}\beta^{-\frac{1}{4}}))\delta_{\alpha\beta} \right]_{\alpha,\beta=0}^k, \quad (3-80)$$

$\Phi(z)$ is defined with the use of the Airy function $Ai(z)$:

$$\begin{cases} [(1 - \delta_{\alpha 0}(1 - y_0(z)) - \delta_{\alpha i}(1 + y'_2(z)))\delta_{\alpha\beta} + \delta_{\alpha i}\delta_{\beta 0}y'_0(z) - \delta_{\alpha i}\delta_{\beta 0}y_2(z)]_{\alpha,\beta=0}^k & \text{for } 0 < \arg < \frac{2\pi}{3} \\ [(1 - \delta_{\alpha 0}(1 + y_1(z)) - \delta_{\alpha i}(1 + y'_2(z)))\delta_{\alpha\beta} - \delta_{\alpha i}\delta_{\beta 0}y'_1(z) - \delta_{\alpha i}\delta_{\beta 0}y_2(z)]_{\alpha,\beta=0}^k & \text{for } \frac{2\pi}{3} < \arg < \pi \\ [(1 - \delta_{\alpha 0}(1 + y_2(z)) - \delta_{\alpha i}(1 - y'_1(z)))\delta_{\alpha\beta} - \delta_{\alpha i}\delta_{\beta 0}y'_2(z) + \delta_{\alpha i}\delta_{\beta 0}y_1(z)]_{\alpha,\beta=0}^k & \text{for } -\pi < \arg < -\frac{2\pi}{3} \\ [(1 - \delta_{\alpha 0}(1 - y_0(z)) - \delta_{\alpha i}(1 - y'_1(z)))\delta_{\alpha\beta} + \delta_{\alpha i}\delta_{\beta 0}y'_0(z) + \delta_{\alpha i}\delta_{\beta 0}y_1(z)]_{\alpha,\beta=0}^k & \text{for } -\frac{2\pi}{3} < \arg < 0 \end{cases} \quad (3-81)$$

where

$$y_\alpha = e^{\frac{2\alpha i\pi}{3}} Ai(e^{\frac{2\alpha i\pi}{3}} z) \quad (3-82)$$

for $\alpha = 0, 1, 2$ and D is the diagonal matrix

$$D(z) = \left[(1 - \delta_{\alpha 0}(1 - e^{\frac{N}{2}(\lambda_0(z) - \lambda_i(z))}) - \delta_{\alpha i}(1 - e^{-\frac{N}{2}(\lambda_0(z) - \lambda_i(z))}))\delta_{\alpha\beta} \right]_{\alpha,\beta=0}^k. \quad (3-83)$$

3.2.5 Final transformation

We now have a parametrix away from the branch points ($M(z)$) and in their neighborhood ($P(z)$). It remains to put them together in a unique function defined on the whole complex plane: this is the last transformation.

Define the matrix

$$\begin{aligned} R(z) &= S(z)M(z)^{-1} \quad \text{for } z \text{ outside the disks } D(z_i, r), i = 1, \dots, 2k \\ R(z) &= S(z)P(z)^{-1} \quad \text{for } z \text{ inside the disks } D(z_i, r), i = 1, \dots, 2k. \end{aligned} \quad (3-84)$$

One can easily check that:

$$R(z) = \mathbf{I} + O\left(\frac{1}{N(|z| + 1)}\right) \quad (3-85)$$

as $N \rightarrow \infty$ uniformly for $z \in \mathbb{C} \setminus \left(\bigcup_{i=1}^{2k} \partial D(z_i, r)\right)$ and

$$R'(z) = O\left(\frac{1}{N(|z| + 1)}\right). \quad (3-86)$$

It follows the important relation:

$$R^{-1}(x_2)R(x_1) = I + O\left(\frac{x_1 - x_2}{N}\right)$$

(3-87)

needed to study the behavior of the Kernel in the different regimes.

3.3 Mean density of particles

With the use of the kernel K_N , one can state more precisely the role played by the spectral curve. It follows from:

Theorem 3.2 *The limiting mean density of eigenvalues of the random matrix*

$$\rho(x) := \lim_{N \rightarrow \infty} \frac{K_N(x, x)}{N} \quad (3-88)$$

exists, is supported by the real cuts $\{[z_{2i-1}, z_{2i}]\}_{i=1}^k$ and is expressed in terms of the solution ξ_0 of the algebraic equation

$$\mathcal{E}(x, \xi) = 0 \quad (3-89)$$

by

$$\rho(x) = \frac{1}{\pi} |\operatorname{Im} \xi_0(x)|. \quad (3-90)$$

It is analytic on the cuts and vanishes like a square root at the edges z_i .

proof:

Let us consider $x \in]z_{2i-1}, z_{2i}[$ for any $i = 1, \dots, k$, assuming that x does not belong to any $D(z_j, r)$. Thus, Eq. (3-84) and Eq. (3-87) imply that

$$S_+^{-1}(y)S_+(x) = \mathbf{I} + O(x - y) \quad (3-91)$$

uniformly in N when $y \rightarrow x$. The expression of the kernel in terms of $S(x)$, Eq. (3-46), gives:

$$\begin{aligned} K_N(x, y) &= \frac{e^{N[h(y)-h(x)]}}{2i\pi(x-y)} \Omega_{1,S,i}(y) [\mathbf{I} + O(x-y)] \Omega_{2,S,i}^t(x) \\ &= e^{N[h(y)-h(x)]} \left[\frac{\sin(N \operatorname{Im}(\lambda_{0+}(x) - \lambda_{0+}(y)))}{\pi(x-y)} + O(1) \right] \end{aligned} \quad (3-92)$$

uniformly in N . Remember that, by definition,

$$\rho(x) = \frac{1}{\pi} \frac{d \operatorname{Im} \lambda_{0+}(x)}{dx}. \quad (3-93)$$

Thus, we get

$$K_N(x, x) = N\rho(x) + O(1), \quad (3-94)$$

proving the first part of the theorem. For x outside of the cuts (or at a branch point), we easily see that

$$\lim_{N \rightarrow \infty} \frac{K_N(x, x)}{N} = 0 \quad (3-95)$$

because of the ordering of the λ functions (see th.3.1). \square

3.4 Behavior in the Bulk

Let us first study the asymptotic of the kernel inside the support of eigenvalues. We recover the usual sine-kernel:

Theorem 3.3 *For any $x \in \bigcup_{i=1}^k]z_{2i-1}, z_{2i}[$ and any $(u, v) \in \mathbb{R}^2$, one has*

$$\boxed{\lim_{N \rightarrow \infty} \frac{1}{N\rho(x)} \widehat{K}_N \left(x + \frac{u}{N\rho(x)}, x + \frac{v}{N\rho(x)} \right) = \frac{\sin \pi(u-v)}{\pi(u-v)}} \quad (3-96)$$

where we use the rescaled kernel:

$$\widehat{K}_N(x, y) = e^{N(h(x)-h(y))} K_N(x, y). \quad (3-97)$$

proof:

Consider $x \in]z_{2i-1}, z_{2i}[$ for any $i = 1, \dots, k$ and define

$$X = x + \frac{u}{N\rho(x)} \quad \text{and} \quad Y = x + \frac{v}{N\rho(x)}. \quad (3-98)$$

Then, there exists $t = x + O\left(\frac{1}{N}\right)$ such that

$$N\text{Im}(\lambda_{0+}(X) - \lambda_{0+}(Y)) = \frac{\pi(u-v)\rho(t)}{\rho(x)}. \quad (3-99)$$

From Eq. (3-92), it follows

$$\begin{aligned} \frac{\widehat{K}_N(X, Y)}{N\rho(x)} &= \frac{\sin(N\text{Im}(\lambda_{0+}(X) - \lambda_{0+}(Y)))}{\pi(u-v)} + O\left(\frac{1}{N}\right) \\ &= \frac{\sin(\pi(u-v))}{\pi(u-v)} + O\left(\frac{1}{N}\right). \end{aligned} \quad (3-100)$$

□

3.5 Behavior at the edge

Let us now study the behavior of the kernel at the edge of the spectrum, i.e. near any branch point of the spectral curve.

Theorem 3.4 For any $i = 1, \dots, 2k$ and every $(u, v) \in \mathbb{R}^2$, the rescaled kernel satisfies:

$$\lim_{N \rightarrow \infty} \frac{1}{(\rho_i N)^{\frac{2}{3}}} \widehat{K}_N \left(z_i + (-1)^i \frac{u}{(\rho_i N)^{\frac{2}{3}}}, z_i + (-1)^i \frac{v}{(\rho_i N)^{\frac{2}{3}}} \right) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u - v}, \quad (3-101)$$

where ρ_i is a constant characterizing the behavior of ρ near the branch point z_i :

$$\rho(x) = \frac{\rho_i}{\pi} |x - z_i|^{\frac{1}{2}} (1 + o(1)) \quad (3-102)$$

when $x \rightarrow z_i$.

proof:

Let us consider an arbitrary branch point⁶ z_{2i} and define

$$X := z_{2i} + \frac{u}{(\rho_{2i} N)^{\frac{2}{3}}} \quad \text{and} \quad Y := z_{2i} + \frac{v}{(\rho_{2i} N)^{\frac{2}{3}}} \quad (3-103)$$

for any $(u, v) \in]0, \infty[^2$ so that, for N large enough, X and Y lie inside the circle $D(z_{2i}, r)$ on the cut. The rescaled kernel can thus be written:

$$\widehat{K}_N(X, Y) = \frac{1}{2i\pi(X - Y)} \Omega_{1,S,i}(Y) S_+^{-1}(Y) S_+(X) \Omega_{2,S,i}^t(X) \quad (3-104)$$

with the notations of Eq. (3-46) and

$$S_+(X) = R(X) P_+(X) = R(X) \widetilde{P}_+(X) \quad (3-105)$$

since we are in the region II. Using the form 3-77 of \widetilde{P} , one gets:

$$S_+(X) = R(X) E_N(X) \Phi_+ \left(N^{\frac{2}{3}} \beta_i(X) \right) D_{i,+}(X) \quad (3-106)$$

where

$$D_{i,+}(X) = \text{diag} \left(e^{Ni\text{Im}[\lambda_{0+}(X)]}, \overbrace{1, \dots, 1}^{i-1}, e^{-Ni\text{Im}[\lambda_{0+}(X)]}, \overbrace{1, \dots, 1}^{k-i} \right). \quad (3-107)$$

It follows

$$\frac{1}{(\rho_{2i} N)^{\frac{2}{3}}} \widehat{K}_N(X, Y) = \frac{1}{2i\pi(u - v)} \widetilde{D}_i \Phi_+^{-1} \left(N^{\frac{2}{3}} \beta_i(Y) \right) E_N^{-1}(Y) R^{-1}(Y) R(X) E_N(X) \Phi_+ \left(N^{\frac{2}{3}} \beta_i(X) \right) \widehat{D}_i^t \quad (3-108)$$

⁶We consider here a branch point on the right side of a cut but we could do exactly the same derivation for the other end point of the cut z_{2i-1} .

where we use the vector of size $k + 1$:

$$\tilde{D}_i := \begin{bmatrix} -1 & \overbrace{0 \dots 0}^{i-1} & 1 & \overbrace{0 \dots 0}^{k-i} \end{bmatrix} \quad (3-109)$$

and

$$\hat{D}_i := \begin{bmatrix} 1 & \overbrace{0 \dots 0}^{i-1} & 1 & \overbrace{0 \dots 0}^{k-i} \end{bmatrix}. \quad (3-110)$$

Remember that $\beta_i(z_{2i}) = 0$ and $\beta'_i(z_{2i}) = \rho_{2i}^{\frac{2}{3}}$. Thus

$$\Phi_+ \left(N^{\frac{2}{3}} \beta_i(X) \right) \rightarrow \Phi_+(u) \quad \text{when } N \rightarrow \infty \quad (3-111)$$

and

$$\lim_{N \rightarrow \infty} \Phi_+ \left(N^{\frac{2}{3}} \beta_i(X) \right) \hat{D}_i^t = \begin{bmatrix} y_0(u) & \overbrace{0 \dots 0}^{i-1} & y'_0(u) & \overbrace{0 \dots 0}^{k-i} \end{bmatrix}^t \quad (3-112)$$

as well as

$$\lim_{N \rightarrow \infty} \tilde{D}_i \Phi_+^{-1} \left(N^{\frac{2}{3}} \beta_i(Y) \right) = -2i\pi \begin{bmatrix} -y'_0(u) & \overbrace{0 \dots 0}^{i-1} & y_0(u) & \overbrace{0 \dots 0}^{k-i} \end{bmatrix} \quad (3-113)$$

with the function y_0 defined by Eq. (3-82) in terms of the Airy function.

Since one knows the behavior of E_N , E_N^{-1} and R when $N \rightarrow \infty$:

$$\begin{aligned} E_N(X) &= O\left(N^{\frac{1}{6}}\right) \\ E_N^{-1}(Y) &= O\left(N^{\frac{1}{6}}\right) \\ E_N^{-1}(Y)E_N(X) &= I + O\left(N^{-\frac{1}{3}}\right) \\ R^{-1}(Y)R(X) &= I + O\left(N^{-\frac{5}{3}}\right) \end{aligned} \quad (3-114)$$

we finally prove that

$$\lim_{N \rightarrow \infty} \frac{1}{(\rho_{2i}N)^{\frac{2}{3}}} \hat{K}_N(X, Y) = \frac{y_0(u)y'_0(v) - y'_0(u)y_0(v)}{u - v} \quad (3-115)$$

which gives the theorem for $u, v > 0$.

For the other signs of u and v , we can derive the result in the same way by using the appropriate expression of the kernel as well as the following definition of the h function outside of the cut. \square

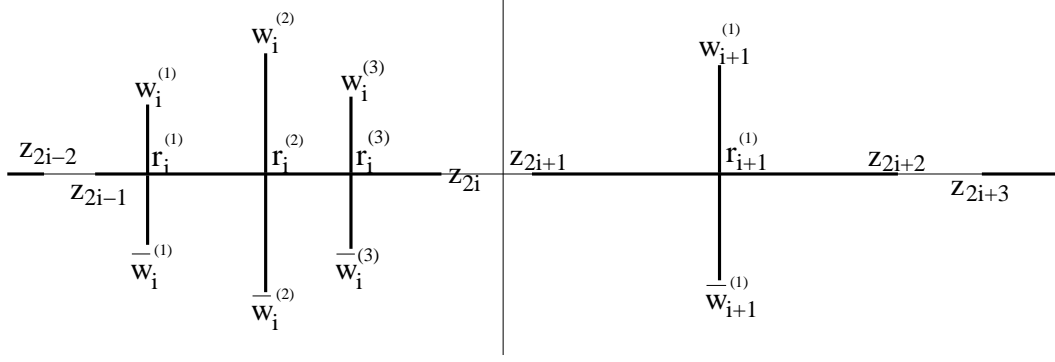


Figure 4: Cuts of the ξ functions in the x -plane

4 Intermediate state

We consider here one intermediate configuration where two or more segment have already merged.

It corresponds to having at least two non-real complex conjugate branch points.

4.1 Study of the spectral curve

We consider an intermediate time in the evolution of the Brownian motions. The spectral curve remains the same:

$$\mathcal{E}(x, y) = 0 \quad (4-1)$$

but the value of the coefficients makes its structure different through the position of the branch points.

Let us consider the following case (which is generic). The $2k$ branch points, solution of the equation in y

$$1 = \sum_{i=1}^k \frac{n_i}{N (y - a_i(t))^2} \quad (4-2)$$

are such that

- there are $2l$ real branch points z_i with $i = 1, \dots, 2l$ and $0 < l < k$;
- for $i = 1, \dots, l$, there are $0 \leq b_i$ pairs of distinct non-real complex conjugated branch points $\left(w_i^{(m)}, \bar{w}_i^{(m)}\right)_{m=1}^{b_i}$ linked by a cut $\Gamma_i^{(m)}$ crossing the real axis in a unique point $r_i^{(m)}$ satisfying:

$$\forall i, z_{2i-1} < r_i^{(1)} < r_i^{(2)} < \dots < r_i^{(b_i)} < z_{2i}. \quad (4-3)$$

Remark 4.1 One obviously has the relation:

$$k = l + \sum_{i=1}^l b_i. \quad (4-4)$$

and we use a double index notation to sort the filling fraction:

$$n_i^{(m)} := n_{\sum_{j=1}^{i-1} b_j + m + i} \quad (4-5)$$

and the eigenvalues:

$$a_i^{(m)} := a_{\sum_{j=1}^{i-1} b_j + m + i}. \quad (4-6)$$

Let us now study the properties of the spectral curve as we did in the preceding section. We are particularly interested in its sheeted structure.

A generic point of the complex plane (i.e. not a branch point) has $k + 1$ pre-images on the spectral curve. Let us note these pre-images by $\xi_0(x)$ and $\xi_i^{(m)}(x)$ with $i = 1, \dots, l$ and $m = 0, \dots, b_i$. These different pre-images are characterized by their asymptotic behaviors when $x \rightarrow \infty$:

$$\begin{aligned} \xi_0(x) &= x - \frac{1}{x} + O\left(\frac{1}{x^2}\right), \\ \xi_i^{(m)} &= a_i^{(m)} + \frac{n_i^{(m)}}{N} \frac{1}{x} + O\left(\frac{1}{x^2}\right). \end{aligned} \quad (4-7)$$

From these asymptotic behaviors, one can define the ξ -functions by analytical continuation as follows⁷:

- $\xi_0(x)$ can be analytically continued to $\mathbb{C} \setminus \bigcup_{i=1}^l [z_{2i-1}, z_{2i}]$;
- for $i = 1, \dots, l$ and $m = 0, \dots, b_i$, $\xi_i^{(m)}(x)$ can be analytically continued to $\mathbb{C} \setminus \left(\Gamma_i^{(m)} \cup [r_i^{(m)}, r_i^{(m+1)}] \cup \Gamma_i^{(m+1)} \right)$ where we use the notation:

$$\begin{cases} w_i^{(0)} = \overline{w}_i^{(0)} = r_i^{(0)} := z_{2i-1}, \\ w_i^{(b_i+1)} = \overline{w}_i^{(b_i+1)} = r_i^{(b_i+1)} := z_{2i}. \end{cases} \quad (4-8)$$

It implies the sheet structure described in fig.5⁸:

- the 0-sheet is connected to the (i, m) -sheet along the real cut $[r_i^{(m)}, r_i^{(m+1)}]$ for $i = 1, \dots, l$ and $m = 0, \dots, b_i$ with the jumps:

$$\xi_{0\pm}(x) = \xi_{i\mp}^{(m)}(x) \text{ for } x \in]r_i^{(m)}, r_i^{(m+1)}[; \quad (4-9)$$

⁷For the sake of brevity, I did not include any proof of this sheet structure in the present manuscript. Nevertheless, the simplicity of the spectral curve implies a simple, but long and uninteresting, proof of these facts. One can, for example, follow the time evolution of the sheet structure from $t = 1$ to $t = 0$ through the rational parametrization of the algebraic curve. As such a proof does not involve any new technical element, it is left to the reader as an exercise.

⁸We label the sheets (except the 0-sheet) by two indices (i, m) corresponding to the associated ξ function $\xi_i^{(m)}$.

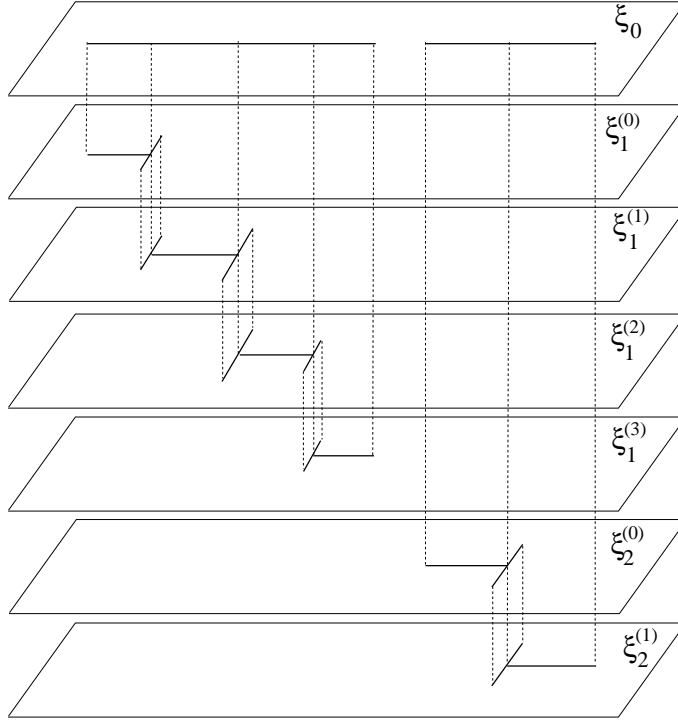


Figure 5: Sheeted structure of the spectral curve for an intermediate step. In this example, there are 4 real branch points and 4 pairs of conjugated ones.

- for $i = 1, \dots, l$ and $m = 1, \dots, b_i - 1$, the (i, m) -sheet is connected to the $(i, m-1)$ - and $(i, m+1)$ -sheets along the imaginary cuts $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$ respectively, with the jumps:

$$\begin{aligned} \xi_{i\pm}^{(m)}(x) &= \xi_{i\mp}^{(m-1)}(x) \text{ for } x \in \Gamma_i^{(m)}, \\ \xi_{i\pm}^{(m)}(x) &= \xi_{i\mp}^{(m+1)}(x) \text{ for } x \in \Gamma_i^{(m+1)}. \end{aligned} \quad (4-10)$$

We finally define the associated λ functions as in the preceding section:

Definition 4.1 *Let the function $\lambda_i^{(m)}$ be the primitives of $\xi_i^{(m)}$ defined by the conditions:*

$$\lambda_0(z_{2l}) = 0, \quad (4-11)$$

$$\lambda_{i+}^{(0)}(z_{2i-1}) = \lambda_{0-}(z_{2i-1}), \quad (4-12)$$

$$\lambda_i^{(b_i)}(z_{2i}) = \lambda_{0+}(z_{2i}) \quad (4-13)$$

and

$$\lambda_{i+}^{(m)}(w_i^{(m+1)}) = \lambda_{i-}^{(m+1)}(w_i^{(m+1)}) \quad (4-14)$$

for $m = 1, \dots, b_i - 1$.

One can easily prove that they satisfy the following properties:

Lemma 4.1 λ_0 is analytic in $\mathbb{C} \setminus \bigcup_i [z_{2i-1}, z_{2i}]$ and $\lambda_i^{(m)}$ is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{i-1} [z_{2j-1}, z_{2j}] \cup [z_{2i-1}, r_i^{(m)}] \cup \Gamma_i^{(m)}$ for $i = 1, \dots, k$ and $m = 1, \dots, b_i$.

The jumps and asymptotics of these λ -functions are given by:

- When $z \rightarrow \infty$:

$$\lambda_0 = \frac{z^2}{2} - \ln z + l_0 + O\left(\frac{1}{z^2}\right) \quad (4-15)$$

and, for $i \neq 0$:

$$\lambda_i^{(m)} = a_i^{(m)} z + \frac{n_i^{(m)}}{N} \ln z + l_i^{(m)} + O\left(\frac{1}{z^2}\right) \quad (4-16)$$

where $l_i^{(m)}$'s are some constants chosen to satisfy the constraints (4-11) – (4-14).

- They follow the jumps:

$$\begin{aligned} \lambda_{0+}(x) - \lambda_{0-}(x) &= -2i\pi \sum_{j=i+1}^l \sum_{m=1}^{b_j} \frac{n_j^{(m)}}{N}, & x \in]z_{2i-2}, z_{2i-1}[\\ \lambda_{0+}(x) - \lambda_{0-}(x) &= -2i\pi, & x \in]-\infty, z_1[. \end{aligned} \quad (4-17)$$

For $i \in [1, l]$

$$\begin{aligned} \lambda_{i+}^{(m)}(x) - \lambda_{i-}^{(m)}(x) &= 2i\pi \frac{n_i^{(m)}}{N}, & x \in]-\infty, r_i^{(m)}[\\ \lambda_{0+}(x) - \lambda_{l-}^{(b_l)}(x) &= 0, \quad \lambda_{0-}(x) - \lambda_{l+}^{(b_l)}(x) = 0, & x \in]z_{2l-1}, z_{2l}[\end{aligned} \quad (4-18)$$

For $i \in [1, l-1]$

$$\lambda_{0+}(x) - \lambda_{i-}^{(m)}(x) = 0, \quad x \in [r_i^{(m)}, r_i^{(m+1)}] \quad (4-19)$$

and

$$\lambda_{i\pm}^{(m)}(x) = \lambda_{i\mp}^{(m+1)}(x), \quad x \in \Gamma_i^{(m)+} \quad (4-20)$$

where $\Gamma_i^{(m)+}$ denotes the part of $\Gamma_i^{(m)}$ lying inside the upper half plane.

4.1.1 Study of the sign of $\operatorname{Re}(\lambda_i^{(m)} - \lambda_j^{(h)})$

It is very important to know the sign of $\operatorname{Re}(\lambda_i^{(m)} - \lambda_j^{(h)})$ for $(i, m) \neq (j, h)$ in order to study the asymptotic behavior of the jump matrices in the Riemann-Hilbert problem we

want to solve. For this purpose, it is convenient to draw the curves where $\text{Re}(\lambda_i^{(m)}) = \text{Re}(\lambda_j^{(h)})$. Let us build them step by step.

First of all, from the behavior of the λ -functions near the branch points, one can state the following results:

- From any branch point z_{2i-1} go three curves at equal angles $\frac{2\pi}{3}$ where $\text{Re}(\lambda_0) = \text{Re}(\lambda_i^{(0)})$, one of them being $[z_{2i-1}, r_i^{(1)}]$.
- From any branch point z_{2i} go three curves at equal angles $\frac{2\pi}{3}$ where $\text{Re}(\lambda_0) = \text{Re}(\lambda_i^{(b_i)})$ one of them being $[r_i^{(b_i)}, z_{2i}]$.
- From any branch point $w_i^{(m)}$ (resp. $\bar{w}_i^{(m)}$) go three curves at equal angles $\frac{2\pi}{3}$ where $\text{Re}(\lambda_i^{(m-1)}) = \text{Re}(\lambda_i^{(m)})$, one of them being $[w_i^{(m)}, i\infty[$ (resp. $[\bar{w}_i^{(m)}, -i\infty[$). Let us denote $\Gamma_{i,R}^{(m+)}$ and $\Gamma_{i,L}^{(m+)}$ (resp. $\Gamma_{i,R}^{(m-)}$ and $\Gamma_{i,L}^{(m-)}$) the two other curve going from $w_i^{(m)}$ (resp. $\bar{w}_i^{(m)}$), the first one lying to the right hand side of the curve $\Gamma_i^{(m)}$ (see fig. 6). The curves $\Gamma_{i,R}^{(m+)}$ and $\Gamma_{i,R}^{(m-)}$ (resp. $\Gamma_{i,L}^{(m+)}$ and $\Gamma_{i,L}^{(m-)}$) intersect in the real line in $x_{i,R}^{(m)}$ (resp. $x_{i,L}^{(m)}$). On the analytical continuation $\bar{\Gamma}_{i,R}^{(m\mp)}$ (resp. $\bar{\Gamma}_{i,L}^{(m\mp)}$) of $\Gamma_{i,R}^{(m\pm)}$ (resp. $\Gamma_{i,L}^{(m\pm)}$) beyond $x_{i,R}^{(m)}$ (resp. $x_{i,L}^{(m)}$), one has $\text{Re}(\lambda_0 - \lambda_i^{(m-1)}) = 0$ (resp. $\text{Re}(\lambda_0 - \lambda_i^{(m)}) = 0$) because of the cut on the real line.

We have investigated the curves $\text{Re}(\lambda_i^{(m)} - \lambda_j^{(h)})$ near the branch points, let us now look at their behavior away from them. Many different configuration can occur, depending on the relative positions of the $x_{i,R}^{(m)}$'s, the $x_{i,L}^{(m)}$'s, the $r_i^{(m)}$'s and the real branch points z_i . In order to limit the size of this paper, we restrict our study to the cases where $r_i^{(m)} < x_{i,R}^{(m)} < r_i^{(m+1)}$ and $r_i^{(m)} < x_{i,L}^{(m+1)} < r_i^{(m+1)}$. Nevertheless, the RH problem corresponding to the other possible configurations can be solved with the same method and the same transformations even if we do not give a proof here (the study if the signs of $\text{Re}(\lambda_i^{(m)} - \lambda_j^{(h)})$ is a little bit more subtle but does not present any particular difficulty).

Note also that the curves $\text{Re}(\lambda_i^{(m-1)}) = \text{Re}(\lambda_i^{(m)})$ and $\text{Re}(\lambda_i^{(m+1)}) = \text{Re}(\lambda_i^{(m)})$ cross in two points symmetric with respect to the real axis. On the line $D_i^{(m+\frac{1}{2})}$ joining these two points, one has $\text{Re}(\lambda_i^{(m-1)}) = \text{Re}(\lambda_i^{(m+1)})$.

With this assumption, between $r_i^{(m)}$ and $r_i^{(m+1)}$, there are two different possible configurations:

- In the first case:

$$r_i^{(m)} < x_{i,R}^{(m)} < x_{i,L}^{(m+1)} < r_i^{(m+1)}. \quad (4-21)$$

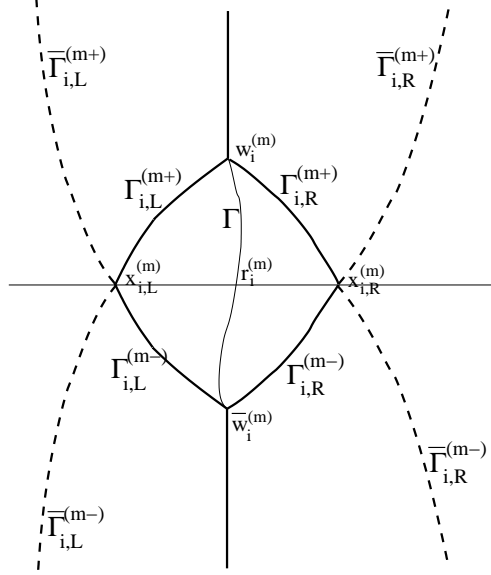


Figure 6: Zoom near the vertical cut $\Gamma_i^{(m)}$. The thick solid lines are the curves where $\text{Re}(\lambda_i^{(m)} - \lambda_i^{(m-1)}) = 0$. On the dashed lines $\bar{\Gamma}_{iR}^{m\pm}$ (resp. $\bar{\Gamma}_{iL}^{m\pm}$), one has $\text{Re}(\lambda_0 - \lambda_i^{(m-1)}) = 0$ (resp. $\text{Re}(\lambda_i^{(m)} - \lambda_0) = 0$).

The respective weights of $\text{Re}(\lambda_i^{m-1}(x))$, $\text{Re}(\lambda_i^m(x))$, $\text{Re}(\lambda_i^{m+1}(x))$ and $\text{Re}(\lambda_0^{(0)}(x))$ for x between $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$ are depicted in fig.7.

- In the second case:

$$r_i^{(m)} < x_{i,L}^{(m+1)} < x_{i,R}^{(m)} < r_i^{(m+1)}. \quad (4-22)$$

The respective weights of $\text{Re}(\lambda_i^{m-1}(x))$, $\text{Re}(\lambda_i^m(x))$, $\text{Re}(\lambda_i^{m+1}(x))$ and $\text{Re}(\lambda_0^{(0)}(x))$ for x between $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$ are depicted in fig.8.

In both configuration one has the following result:

Lemma 4.2 *For x between $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$:*

$$\lambda_j^{(h)}(x) > \lambda_0(x) \quad (4-23)$$

for $(j, h) \neq (i, m-1), (i, m), (i, m+1)$.

4.2 Riemann-Hilbert analysis

We now solve the Riemann Hilbert problem presented in the preceding section for this particular value of the external matrix eigenvalues a_i by performing a sequence of transformation leading to a simple problem when $N \rightarrow \infty$.

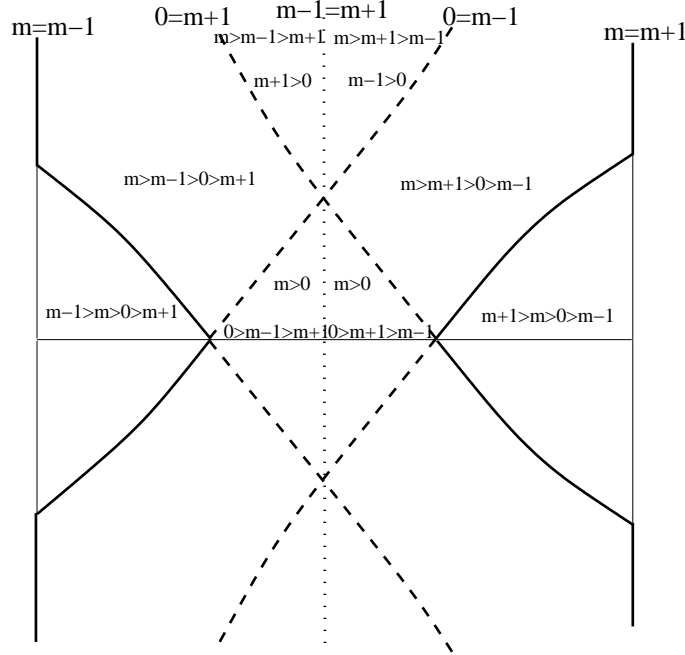


Figure 7: Study of the sign of $\text{Re}(\lambda_j^{(h)} - \lambda_j^{(h')})$ for the first case. For readability, $m > m+1 > m-1 > 0$ stands for $\text{Re}(\lambda_i^m(x)) > \text{Re}(\lambda_i^{m+1}(x)) > \text{Re}(\lambda_i^{m-1}(x)) > \text{Re}(\lambda_0^0(x))$.

For convenience, we use a double index notation for the matrices in this section: considering an arbitrary $k+1 \times k+1$ matrix A , we note its matrix elements as follows:

$$[A]_{(i,m);(i',m')} := [A]_{\sum_{j=1}^{i-1} b_j + m + i, \sum_{j=1}^{i'-1} b_j + m' + i'} \quad (4-24)$$

for $i = 1, \dots, l$ and $m = 0, \dots, b_i$. The first line (or column) is denoted by the pair $(0, 0)$. This notation coincides with the notation used to sort the branch points and filling fraction as well as the $\xi_i^{(m)}$ and $\lambda_i^{(m)}$ functions: the lines are sorted in i groups of b_i elements corresponding to the i real cuts $[z_{2i-1}, z_{2i}]$ shared by $b_i + 1$ sheets.

With these notations, let us remember the RH problem to solve. We are looking for a $k+1 \times k+1$ matrix $Y(x)$ satisfying the constraints:

- Y is analytic on $\mathbb{C} \setminus \mathbb{R}$;
- for $x \in \mathbb{R}$, one has the jumps

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega_1^{(0)}(x) & \omega_1^{(1)}(x) & \dots & \omega_l^{(b_l)}(x) \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad (4-25)$$

and

$$\Omega_{2,YT} := \text{diag} \left(e^{N\left(\lambda_0 - \frac{x^2}{2}\right)}, e^{N\left(\lambda_1^{(0)} - \frac{n_1^{(0)}}{N}x\right)}, e^{N\left(\lambda_1^{(1)} - \frac{n_1^{(1)}}{N}x\right)}, \dots, e^{N\left(\lambda_l^{(b_l)} - \frac{n_l^{(b_l)}}{N}x\right)} \right). \quad (4-29)$$

Then one can check that it is solution of the Riemann-Hilbert problem:

- T is analytic on $x \in \mathbb{C} \setminus \mathbb{R} \bigcup_{i=1}^l \bigcup_{m=1}^{b_i} [w_i^{(m)}, \bar{w}_i^{(m)}]$;
- T has the jumps

$$T_+(x) = T_-(x)j_T(x) \quad (4-30)$$

where

$$j_T(x) = \begin{pmatrix} e^{N(\lambda_{0+}(x) - \lambda_{0-}(x))} & e^{N(\lambda_{1+}^{(0)}(x) - \lambda_{0-}(x))} & e^{N(\lambda_{1+}^{(1)}(x) - \lambda_{0-}(x))} & \dots & e^{N(\lambda_{l+}^{(b_l)}(x) - \lambda_{0-}(x))} \\ 0 & e^{N(\lambda_{1+}^{(0)}(x) - \lambda_{1-}^{(0)}(x))} & 0 & \dots & 0 \\ 0 & 0 & e^{N(\lambda_{1+}^{(1)}(x) - \lambda_{1-}^{(1)}(x))} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & e^{N(\lambda_{l+}^{(b_l)}(x) - \lambda_{l-}^{(b_l)}(x))} \end{pmatrix} \quad (4-31)$$

for $x \in \mathbb{R}$ and

$$[j_T(x)]_{(j,h);(j',h')} = \delta_{(j,h);(j',h')} \left[1 + (\delta_{(j,h);(i,m)} + \delta_{(j,h);(i,m+1)}) \left(e^{N(\lambda_{j+}^{(h)} - \lambda_{j-}^{(h)})} - 1 \right) \right] \quad (4-32)$$

for $x \in [w_i^{(m+1)}, \bar{w}_i^{(m+1)}]$ with $i = 1, \dots, l$ and $m = 0, \dots, b_i - 1$ ⁹.

- when $x \rightarrow \infty$, one has the asymptotic behavior:

$$T(x) = \left(\mathbf{I} + O\left(\frac{1}{x}\right) \right). \quad (4-33)$$

Note that one can factorize the jump matrix $j_T(x)$ depending on the cut on which x lies:

For $x \in [r_i^{(m)}, r_i^{(m+1)}]$ with $i = 1, \dots, l$ and $m = 0, \dots, b_i$, all the diagonal terms of j_T reduce to 1 except the first one and the term $[(i, m); (i, m)]$ whereas the term

⁹It simply means that the jump matrix is the identity for $x \in [w_i^{(m+1)}, \bar{w}_i^{(m+1)}]$ except for the (i, m) 'th and $(i, m+1)$ 'th diagonal terms which are equal to $e^{N(\lambda_{i+}^{(m)} - \lambda_{i-}^{(m)})}$ and $e^{N(\lambda_{i+}^{(m+1)} - \lambda_{i-}^{(m+1)})}$ respectively.

$[(0, 0); (i, m)]$ (i.e. the (i, m) 'th term of the first line) also goes to 1^{10} :

$$\begin{pmatrix} e^{N(\lambda_{0+}-\lambda_{0-})} & e^{N(\lambda_{1+}^{(0)}-\lambda_{0-})} & \dots & e^{N(\lambda_{i+}^{(m-1)}-\lambda_{0-})} & 1 & e^{N(\lambda_{i+}^{(m+1)}-\lambda_{0-})} & \dots & e^{N(\lambda_{i+}^{(b_l)}-\lambda_{0-})} \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & e^{N(\lambda_{i+}^{(m)}-\lambda_{i-}^{(m)})} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (4-34)$$

For $x \in \mathbb{R} \setminus \bigcup_i [z_{2i-1}, z_{2i}]$, all the diagonal terms of j_T reduce to 1 and it takes the form

$$j_T = \begin{pmatrix} 1 & e^{N(\lambda_{1+}^{(0)}(x)-\lambda_{0-}(x))} & e^{N(\lambda_{1+}^{(1)}(x)-\lambda_{0-}(x))} & \dots & e^{N(\lambda_{i+}^{(b_l)}(x)-\lambda_{0-}(x))} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (4-35)$$

The kernel can be written:

$$K_N(x, y) = \frac{e^{N\left(\frac{x^2}{4} - \frac{y^2}{4}\right)}}{2i\pi(x-y)} \Omega_{1,T}(y) T_+^{-1}(y) T_+(x) \Omega_{2,T}^t(x) \quad (4-36)$$

where

$$\Omega_{1,T}(x) = [0 \ e^{N\lambda_{1+}^{(0)}(x)} \ \dots \ e^{N\lambda_{i+}^{(b_l)}(x)}] \quad (4-37)$$

and

$$\Omega_{2,T}(x) = [e^{-N\lambda_{0+}(x)} \ 0 \ \dots \ 0]. \quad (4-38)$$

4.2.2 Second transformation, global opening of lenses

We now proceed to the second transformation allowing to cancel the exponentially increasing coefficients of the jump matrix j_T . This transformation was introduced in [4], for the particular case of two imaginary branch points $i = 1$ and $b_1 = 1$. We generalize this procedure using intensively the study of the spectral curve presented in the preceding section.

For this purpose, we define a contour $\Sigma = \bigcup_{i=1}^l \Sigma_i$ composed of l connected components Σ_i build as follows (see figures 7, 8 and 9):

¹⁰We show here the form of j_T for $m \neq 0, b_i$. These two cases take exactly the same kind of form.

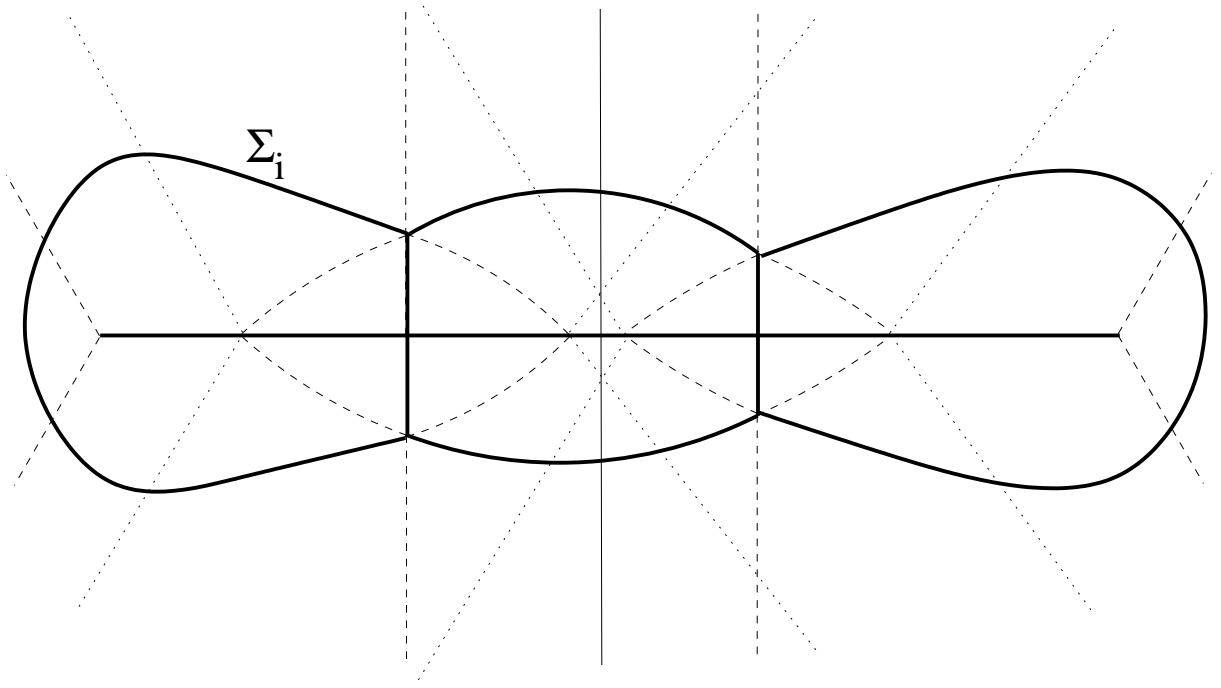


Figure 9: Example of contour Σ used for the global opening of lenses. Here, $b_i = 2$ and we are in the first case where $x_{i,R}^{(1)} < x_{i,L}^{(2)}$.

- Σ_i is a closed curved encircling the cut $[z_{2i-1}, z_{2i}]$ cutting the real axis in x_{2i-1} and x_{2i} satisfying:

$$x_{2i-2} < x_{2i-1} < z_{2i-1} < z_{2i} < x_{2i}; \quad (4-39)$$

- Σ_i intersects the line $r_i^{(m)} + i\mathbb{R}$ in $w_i^{(m)}$ and $\overline{w}_i^{(m)}$ for $m = 1, \dots, b_i$;
- in a neighborhood of $w_i^{(h)}$ and $\overline{w}_i^{(m)}$, Σ_i is the analytic continuation of the curves $\text{Re}(\lambda_i^{(m)}) = \text{Re}(\lambda_i^{(m-1)})$;
- in the half-plane to the left (resp. to the right) of $r_i^{(m)} + i\mathbb{R}$, Σ lies in the region $\text{Re}(\lambda_i^{(m-1)}) > \text{Re}(\lambda_i^{(m)})$ (resp. $\text{Re}(\lambda_i^{(m-1)}) < \text{Re}(\lambda_i^{(m)})$) for any $m = 1, \dots, b_i$.

Let us now define the second transformation $T \rightarrow U$.

Definition 4.2 Let $U(x)$ be the $k+1 \times k+1$ matrix defined by $U(x) = T(x)J_{UT}(x)$ where J_{UT} is given by

- For x outside Σ , $J_{UT}(x) = \mathbf{I}$;
- For x inside Σ_i and $\text{Re}(x) < r_i(1)$:

$$[J_{UT}(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} - \delta_{(j,h);(i,0)} \delta_{(j',h');(i,1)} e^{N(\lambda_i^{(1)}(x) - \lambda_i^{(0)}(x))}; \quad (4-40)$$

- For x inside Σ_i between $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$:

$$[J_{UT}(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} - \delta_{(j,h);(i,m)} \delta_{(j',h');(i,m-1)} e^{N(\lambda_i^{(m-1)}(x) - \lambda_i^{(m)}(x))} \\ - \delta_{(j,h);(i,m)} \delta_{(j',h');(i,m+1)} e^{N(\lambda_i^{(m+1)}(x) - \lambda_i^{(m)}(x))}; \quad (4-41)$$

- For x inside Σ_i and $\text{Re}(x) > r_i(b_i)$:

$$[J_{UT}(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} - \delta_{(j,h);(i,b_i)} \delta_{(j',h');(i,b_i-1)} e^{N(\lambda_i^{(b_i-1)}(x) - \lambda_i^{(b_i)}(x))}. \quad (4-42)$$

The matrix $U(x)$ is then solution of the Riemann-Hilbert problem:

- $U(x)$ is analytic in $\mathbb{C} \setminus \left(\mathbb{R} \cup \bigcup_{i,m} \Sigma_i^{(m)} \right)$;
- when $x \rightarrow \infty$, one has the asymptotic behavior:

$$U(x) = \left(\mathbf{I} + O\left(\frac{1}{x}\right) \right). \quad (4-43)$$

- One has the jumps $U_+(x) = U_-(x)j_U(x)$ with the jump matrix given by:
+ for $x \in]r_i^{(m)}, r_i^{(m+1)}[$, with $i = 1, \dots, l$ and $m = 0, \dots, b_i$, the jump of the matrix U is the same as jump matrix j_T of Eq. (4-34) except the $(i, m-1)$ 'th and the $(i, m+1)$ 'th terms of the first line which are taken equal to 0:

$$\begin{pmatrix} e^{N(\lambda_{0+}(x) - \lambda_{0-}(x))} & e^{N(\lambda_{1+}^{(0)}(x) - \lambda_{0-}(x))} & \dots & 0 & 1 & 0 & \dots & e^{N(\lambda_{l+}^{(b_l)}(x) - \lambda_{0-}(x))} \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & e^{N(\lambda_{i+}^{(m)}(x) - \lambda_{i-}^{(m)}(x))} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (4-44)$$

+ for $x \in [z_{2i-1}, r_i^{(1)}]$:

$$j_U = \begin{pmatrix} e^{N(\lambda_{0+}(x) - \lambda_{0-}(x))} & 1 & 0 & e^{N(\lambda_{1+}^{(2)}(x) - \lambda_{0-}(x))} & \dots & e^{N(\lambda_{l+}^{(b_l)}(x) - \lambda_{0-}(x))} \\ 0 & e^{N(\lambda_{1+}^{(0)}(x) - \lambda_{1-}^{(0)})} & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (4-45)$$

+ for $x \in [r_i^{(b_i)}, z_{2i}]$, j_U is given by:

$$\begin{pmatrix} e^{N(\lambda_{0+}(x)-\lambda_{0-}(x))} & e^{N(\lambda_{1+}^{(0)}(x)-\lambda_{0-}(x))} & \dots & 0 & 1 & e^{N(\lambda_{i+1+}^{(0)}(x)-\lambda_{0-}(x))} & \dots & e^{N(\lambda_{l+}^{(b_l)}(x)-\lambda_{0-}(x))} \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & 1 & \dots & 0 & \dots & \dots \\ 0 & 0 & \dots & 0 & e^{N(\lambda_{i+}^{(b_i)}(x)-\lambda_{i-}^{(b_i)}(x))} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (4-46)$$

+ for $x \in \mathbb{R} \setminus \cup_i [x_{2i-1}, x_{2i}]$

$$j_U = \begin{pmatrix} 1 & e^{N(\lambda_{1+}^{(0)}(x)-\lambda_{0-}(x))} & e^{N(\lambda_{1+}^{(1)}(x)-\lambda_{0-}(x))} & \dots & e^{N(\lambda_{l+}^{(b_l)}(x)-\lambda_{0-}(x))} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (4-47)$$

+ for $x \in [x_{2i-1}, z_{2i-1}]$, j_U is the same matrix as for $x \in \mathbb{R} \setminus \cup_i [x_{2i-1}, x_{2i}]$ with the term $(i, 1)$ of the first line taken equal to 0.

+ for $x \in [z_{2i}, x_{2i}]$ j_U is the same matrix as for $x \in \mathbb{R} \setminus \cup_i [x_{2i-1}, x_{2i}]$ with the term $(i, b_i - 1)$ of the first line taken equal to 0.

+ for $x \in \Gamma_i^{(m)}$

$$\begin{aligned} [j_U(x)]_{(j,h);(j',h')} &= \delta_{(j,h);(j',h')} \left[1 - \delta_{(j,h);(i,m)} + \delta_{(j,h);(i,m-1)} \left(e^{N(\lambda_{i+}^{(m-1)} - \lambda_{i-}^{(m-1)})} - 1 \right) \right] \\ &\quad + \delta_{(j,h);(i,m)} \delta_{(j',h');(i,m-1)} - \delta_{(j,h);(i,m-1)} \delta_{(j',h');(i,m)} \\ &\quad - \delta_{(j,h);(i,m-1)} \delta_{(j',h');(i,m-2)} e^{N(\lambda_i^{(m-2)} - \lambda_{i-}^{(m-1)})} \\ &\quad + \delta_{(j,h);(i,m)} \left[\delta_{(j',h');(i,m+1)} e^{N(\lambda_i^{(m+1)} - \lambda_{i-}^{(m)})} - \delta_{(j',h');(i,m-2)} e^{N(\lambda_i^{(m-2)} - \lambda_{i+}^{(m)})} \right] \end{aligned} \quad (4-48)$$

It means that the jump matrix reduces to the identity outside the 2×4 block:

$$\begin{array}{c|cccc} & (i, m-2) & (i, m-1) & (i, m) & (i, m+1) \\ \hline (i, m-1) & e^{N(\lambda_i^{(m-2)} - \lambda_{i-}^{(m-1)})} & e^{N(\lambda_{i+}^{(m-1)} - \lambda_{i-}^{(m-1)})} & -1 & 0 \\ (i, m) & e^{N(\lambda_i^{(m-2)} - \lambda_{i+}^{(m)})} & 1 & 0 & e^{N(\lambda_i^{(m+1)} - \lambda_{i-}^{(m)})} \end{array} \quad (4-49)$$

+ for $x \in \Sigma_i$ between $x_{2i-1} + i\mathbb{R}$ and $r_i^{(1)} + i\mathbb{R}$

$$[j_U(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,0)} \delta_{(j',h');(i,1)} e^{N(\lambda_i^{(1)}(x) - \lambda_i^{(0)}(x))}; \quad (4-50)$$

+ for $x \in \Sigma_i$ between $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$:

$$\begin{aligned} [j_U(x)]_{(j,h);(j',h')} &:= \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,m)} \delta_{(j',h');(i,m-1)} e^{N(\lambda_i^{(m-1)}(x) - \lambda_i^{(m)}(x))} \\ &\quad + \delta_{(j,h);(i,m)} \delta_{(j',h');(i,m+1)} e^{N(\lambda_i^{(m+1)}(x) - \lambda_i^{(m)}(x))}; \end{aligned} \quad (4-51)$$

+ for $x \in \Sigma_i$ between $r_i^{(b_i)} + i\mathbb{R}$ and $x_{2i} + i\mathbb{R}$

$$[j_U(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,b_i)} \delta_{(j',h');(i,b_i-1)} e^{N(\lambda_i^{(b_i-1)}(x) - \lambda_i^{(b_i)}(x))}. \quad (4-52)$$

Using the study of the sign of $\text{Re}(\lambda_i^{(m)}) - \text{Re}(\lambda_j^{(h)})$ performed in the beginning of this section, let us check that these jump matrices have no exponentially increasing entries when $N \rightarrow \infty$. For $x \in \Sigma$ and $x \in \left(\mathbb{R} \setminus \bigcup_i [z_{2i-1}, z_{2i}] \right)$, the jump matrices converge to the identity matrix as $N \rightarrow \infty$. For $x \in \Gamma_i^{(m)}$, the jump matrix 4-48 reduces to the identity except the block 4-49 which converges to the block

$$\begin{array}{c|cccc} & (i, m-2) & (i, m-1) & (i, m) & (i, m+1) \\ \hline (i, m-1) & 0 & 0 & -1 & 0 \\ (i, m) & 0 & 1 & 0 & 0 \end{array} \quad (4-53)$$

exactly as it was encountered in the simpler case studied by [4].

Finally, all the non-constant off-diagonal entries of the remaining jump matrices 4-44, 4-45 and 4-46 converge to 0 as $N \rightarrow \infty$. The non-constant diagonal entries have modulus 1 and are rapidly oscillating as $N \rightarrow \infty$. The third transformation aims to turn them to exponentially decreasing terms.

4.2.3 Third transformation

The third transformation aims to cancel the oscillating parts of the jump matrices by opening lenses on $[z_{2i-1}, z_{2i}]$ as in the large time case.

Remark that one can further factorize the matrix $j_U(x)$ for $x \in [r_i^{(m)}, r_i^{(m+1)}]$:

$$j_U(x) = \tilde{j}_{US}(x) j_S(x) \hat{j}_{US}(x) \quad (4-54)$$

where, for $x \in [r_i^{(m)}, r_i^{(m+1)}]$, one has defined the $k+1 \times k+1$ matrices:

$$\begin{aligned} [(j_S)]_{(j,h);(j',h')} &:= (1 - \delta_{(j,h);(i,m)} - \delta_{(j,h);(0,0)}) \delta_{(j,h);(j',h')} + \\ &\quad + \delta_{(j,h);(0,0)} \delta_{(j',h');(i,m)} - \delta_{(j,h);(i,m)} \delta_{(j',h');(0,0)} + \\ &\quad + \delta_{(j,h);(0,0)} \sum_{\substack{(j'', h'') \neq (0,0), (i,m), \\ (i, m-1), (i, m+1)}} \delta_{(j',h');(j'', h'')} e^{N(\lambda_{j'+}^{(h')} - \lambda_{0-})} - \end{aligned}$$

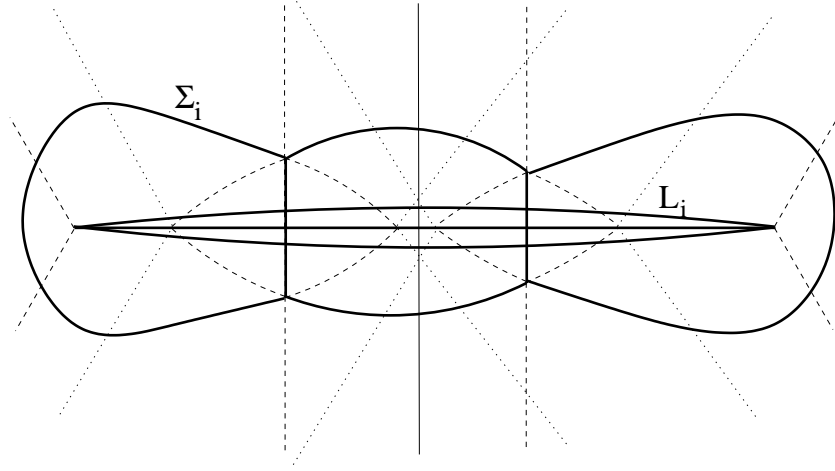


Figure 10: Example of lens for the third transformation.

$$-\delta_{(j,h);(i,m)} \sum_{\substack{(j'',h'') \neq (0,0), (i,m), \\ (i,m-1), (i,m+1)}} \delta_{(j',h');(j'',h'')} e^{N(\lambda_{j'+}^{(h')} - \lambda_{0+})}, \quad (4-55)$$

$$\left[\tilde{j}_{US}(x) \right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,m)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(m)})_-} \quad (4-56)$$

and

$$\left[\hat{j}_{US}(x) \right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,m)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(m)})_+}. \quad (4-57)$$

As in the large time case, we define a set of l lenses L_i with vertices z_{2i} and z_{2i-1} (see fig.10) which do not intersect the contour Σ . Using these lenses, we define the change of variables:

$$S(x) := U(x) \quad (4-58)$$

outside of the lenses and, for x inside the lenses:

$$S(x) := \begin{cases} U(x) J_{US}^{(up)}(x) & \text{in the upper lens region} \\ U(x) J_{US}^{(down)}(x) & \text{in the lower lens region} \end{cases} \quad (4-59)$$

with

$$\begin{cases} \left[J_{US}^{(up)} \right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} - \delta_{(j,h);(i,m)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(m)})} \\ \left[J_{US}^{(down)} \right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,m)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(m)})} \end{cases} \quad (4-60)$$

for x between $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$,

$$\begin{cases} \left[J_{US}^{(up)} \right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} - \delta_{(j,h);(i,0)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(0)})} \\ \left[J_{US}^{(down)} \right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,0)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(0)})} \end{cases} \quad (4-61)$$

for x between z_{2i-1} and $\Gamma_i^{(1)}$ and,

$$\begin{cases} \left[J_{US}^{(up)} \right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} - \delta_{(j,h);(i,b_i)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(b_i)})} \\ \left[J_{US}^{(down)} \right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,b_i)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(b_i)})} \end{cases} \quad (4-62)$$

for x between $\Gamma_i^{(b_i)}$ and z_{2i} .

We finally get the jump matrices for $S(x)$ out of the discontinuities of $U(x)$ thanks to the factorization of the jump matrices for U on the real axis. One has the jumps $S_+(x) = S_-(x)j_S(x)$ with the jump matrix given by:

- for $x \in]r_i^{(m)}, r_i^{(m+1)}[$, with $i = 1, \dots, l$ and $m = 0, \dots, b_i$:

$$\begin{aligned} [(j_S)]_{(j,h);(j',h')} &:= (1 - \delta_{(j,h);(i,m)} - \delta_{(j,h);(0,0)}) \delta_{(j,h);(j',h')} + \delta_{(j,h);(0,0)} \delta_{(j',h');(i,m)} - \delta_{(j,h);(i,m)} \delta_{(j',h');(0,0)} \\ &\quad + \delta_{(j,h);(0,0)} \sum_{\substack{(j'', h'') \neq (0,0), (i,m), \\ (i,m-1), (i,m+1)}} \delta_{(j',h');(j'',h'')} e^{N(\lambda_{j'+}^{(h')} - \lambda_{0-})} \\ &\quad - \delta_{(j,h);(i,m)} \sum_{\substack{(j'', h'') \neq (0,0), (i,m), \\ (i,m-1), (i,m+1)}} \delta_{(j',h');(j'',h'')} e^{N(\lambda_{j'+}^{(h')} - \lambda_{0+})} \end{aligned} \quad (4-63)$$

- for $x \in \mathbb{R} \setminus \cup_i [x_{2i-1}, x_{2i}]$

$$j_S = \begin{pmatrix} 1 & e^{N(\lambda_{1+}^{(0)}(x) - \lambda_{0-}(x))} & e^{N(\lambda_{1+}^{(1)}(x) - \lambda_{0-}(x))} & \dots & e^{N(\lambda_{l+}^{(b_l)}(x) - \lambda_{0-}(x))} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (4-64)$$

- for $x \in [x_{2i-1}, z_{2i-1}]$, j_S is the same matrix as for $x \in \mathbb{R} \setminus \cup_i [x_{2i-1}, x_{2i}]$ with the term $(i, 1)$ of the first line taken equal to 0.
- for $x \in [z_{2i}, x_{2i}]$ j_S is the same matrix as for $x \in \mathbb{R} \setminus \cup_i [x_{2i-1}, x_{2i}]$ with the term $(i, b_i - 1)$ of the first line taken equal to 0.
- For $x \in \Gamma_i^{(m)}$ between $\tilde{x}_{i,up}^{(m)}$ and $w_i^{(m)}$ and between $\bar{w}_i^{(m)}$ and $\tilde{x}_{i,down}^{(m)}$

$$\begin{aligned} [j_S(x)]_{(j,h);(j',h')} &= \delta_{(j,h);(j',h')} \left[1 - \delta_{(j,h);(i,m)} + \delta_{(j,h);(i,m-1)} \left(e^{N(\lambda_{i+}^{(m-1)} - \lambda_{i-}^{(m-1)})} - 1 \right) \right] \\ &\quad + \delta_{(j,h);(i,m)} \delta_{(j',h');(i,m-1)} - \delta_{(j,h);(i,m-1)} \delta_{(j',h');(i,m)} \end{aligned}$$

$$\begin{aligned}
& -\delta_{(j,h);(i,m-1)}\delta_{(j',h');(i,m-2)}e^{N(\lambda_i^{(m-2)}-\lambda_{i-}^{(m-1)})} \\
& +\delta_{(j,h);(i,m)}\left[\delta_{(j',h');(i,m+1)}e^{N(\lambda_i^{(m+1)}-\lambda_{i-}^{(m)})}-\delta_{(j',h');(i,m-2)}e^{N(\lambda_i^{(m-2)}-\lambda_{i+}^{(m)})}\right]
\end{aligned}
\tag{4-65}$$

It means that the jump matrix reduces to the identity outside the 2×4 block:

$$\begin{array}{c|cccc}
& (i, m-2) & (i, m-1) & (i, m) & (i, m+1) \\
\hline
(i, m-1) & e^{N(\lambda_i^{(m-2)}-\lambda_{i-}^{(m-1)})} & e^{N(\lambda_{i+}^{(m-1)}-\lambda_{i-}^{(m-1)})} & -1 & 0 \\
(i, m) & e^{N(\lambda_i^{(m-2)}-\lambda_{i+}^{(m)})} & 1 & 0 & e^{N(\lambda_i^{(m+1)}-\lambda_{i-}^{(m)})}
\end{array}
\tag{4-66}$$

- For $x \in \Gamma_i^{(m)}$ between $r_i^{(m)}$ and $\tilde{x}_{i,up}^{(m)}$:

$$\begin{aligned}
[jS(x)]_{(j,h);(j',h')} &= \delta_{(j,h);(j',h')} \left[1 - \delta_{(j,h);(i,m)} + \delta_{(j,h);(i,m-1)} \left(e^{N(\lambda_{i+}^{(m-1)}-\lambda_{i-}^{(m-1)})} - 1 \right) \right] \\
& -\delta_{(j,h);(i,m-1)}\delta_{(j',h');(0,0)}e^{N(\lambda_0-\lambda_{i-}^{(m-1)})} \\
& +\delta_{(j,h);(i,m)}\delta_{(j',h');(i,m-1)} - \delta_{(j,h);(i,m-1)}\delta_{(j',h');(i,m)} \\
& -\delta_{(j,h);(i,m-1)}\delta_{(j',h');(i,m-2)}e^{N(\lambda_i^{(m-2)}-\lambda_{i-}^{(m-1)})} \\
& +\delta_{(j,h);(i,m)}\left[\delta_{(j',h');(i,m+1)}e^{N(\lambda_i^{(m+1)}-\lambda_{i-}^{(m)})}-\delta_{(j',h');(i,m-2)}e^{N(\lambda_i^{(m-2)}-\lambda_{i+}^{(m)})}\right]
\end{aligned}
\tag{4-67}$$

i.e. this is the same jump matrix as 4-65 except the term $(0, 0)$ of the $(i, m-1)$ 'st line which is equal to $-e^{N(\lambda_0-\lambda_{i-}^{(m-1)})}$ instead of 0.

- For $x \in \Gamma_i^{(m)}$ between $\tilde{x}_{i,down}^{(m)}$ and $r_i^{(m)}$:

$$\begin{aligned}
[jS(x)]_{(j,h);(j',h')} &= \delta_{(j,h);(j',h')} \left[1 - \delta_{(j,h);(i,m)} + \delta_{(j,h);(i,m-1)} \left(e^{N(\lambda_{i+}^{(m-1)}-\lambda_{i-}^{(m-1)})} - 1 \right) \right] \\
& +\delta_{(j,h);(i,m-1)}\delta_{(j',h');(0,0)}e^{N(\lambda_0-\lambda_{i-}^{(m-1)})} \\
& +\delta_{(j,h);(i,m)}\delta_{(j',h');(i,m-1)} - \delta_{(j,h);(i,m-1)}\delta_{(j',h');(i,m)} \\
& -\delta_{(j,h);(i,m-1)}\delta_{(j',h');(i,m-2)}e^{N(\lambda_i^{(m-2)}-\lambda_{i-}^{(m-1)})} \\
& +\delta_{(j,h);(i,m)}\left[\delta_{(j',h');(i,m+1)}e^{N(\lambda_i^{(m+1)}-\lambda_{i-}^{(m)})}-\delta_{(j',h');(i,m-2)}e^{N(\lambda_i^{(m-2)}-\lambda_{i+}^{(m)})}\right]
\end{aligned}
\tag{4-68}$$

i.e. this is the same jump matrix as 4-65 except the term $(0, 0)$ of the $(i, m-1)$ 'st line which is equal to $e^{N(\lambda_0-\lambda_{i-}^{(m-1)})}$ instead of 0.

- for $x \in \Sigma_i$ between $x_{2i-1} + i\mathbb{R}$ and $r_i^{(1)} + i\mathbb{R}$

$$[jS(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,0)}\delta_{(j',h');(i,1)}e^{N(\lambda_i^{(1)}(x)-\lambda_i^{(0)}(x))}; \tag{4-69}$$

- for $x \in \Sigma_i$ between $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$:

$$[j_S(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,m)} \delta_{(j',h');(i,m-1)} e^{N(\lambda_i^{(m-1)}(x) - \lambda_i^{(m)}(x))} + \delta_{(j,h);(i,m)} \delta_{(j',h');(i,m+1)} e^{N(\lambda_i^{(m+1)}(x) - \lambda_i^{(m)}(x))}; \quad (4-70)$$

- for $x \in \Sigma_i$ between $r_i^{(b_i)} + i\mathbb{R}$ and $x_{2i} + i\mathbb{R}$

$$[j_S(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,b_i)} \delta_{(j',h');(i,b_i-1)} e^{N(\lambda_i^{(b_i-1)}(x) - \lambda_i^{(b_i)}(x))}. \quad (4-71)$$

- for $x \in L_i$ between $\Gamma_i^{(m)}$ and $\Gamma_i^{(m+1)}$

$$[J_S]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,m)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(m)})} \quad (4-72)$$

- for $x \in L_i$ between z_{2i-1} and $\Gamma_i^{(1)}$

$$[J_S]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,0)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(0)})} \quad (4-73)$$

- for $x \in L_i$ between $\Gamma_i^{(b_i)}$ and z_{2i}

$$[J_S]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} + \delta_{(j,h);(i,b_i)} \delta_{(j',h');(0,0)} e^{N(\lambda_0 - \lambda_i^{(b_i)})} \quad (4-74)$$

Now, when $N \rightarrow \infty$ all the non-constant terms of the jump matrices tend to 0. We are then left with a model Riemann-Hilbert problem reduced locally to a 2×2 matrix problem similar to the one studied in the preceding section. We can thus use the same parametrix.

4.2.4 Model Riemann-Hilbert problem

We proceed exactly as in the preceding section (large time case) and consider the following RH problem (this is the limiting problem obtained when $N \rightarrow \infty$).

Definition 4.3 *M is solution to the following RH problem:*

- $M : \mathbb{C} \setminus \left(\cup_i [z_{2i-1}, z_{2i}] \cup_{i,m} \Gamma_i^{(m)} \right) \rightarrow \mathbb{C}^{k+1 \times k+1}$ is analytic
- $M(x) = I_{k+1 \times k+1} + O\left(\frac{1}{x}\right)$ when $x \rightarrow \infty$
- $M(x)$ satisfies the jumps $M_+(x) = M_-(x)j_M(x)$ with:

$$[j_M(x)]_{(j,h);(j',h')} = \delta_{(j,h);(j',h')} \left(1 - \delta_{(j,h);(0,0)} - \delta_{(j,h);(i,m)} \right) + \delta_{(j,h);(0,0)} \delta_{(j',h');(i,m)} - \delta_{(j',h');(0,0)} \delta_{(j,h);(i,m)} \quad (4-75)$$

for $x \in]r_i^{(m)}, r_i^{(m+1)}[$ and

$$[j_M(x)]_{(j,h),(j',h')} = \delta_{(j,h),(j',h')} (1 - \delta_{(j,h),(i,m-1)} - \delta_{(j,h),(i,m)}) + \delta_{(j,h),(i,m-1)} \delta_{(j',h'),(i,m)} - \delta_{(j',h'),(i,m-1)} \delta_{(j,h),(i,m)} \quad (4-76)$$

for $x \in \Gamma_i^{(m)}$.

We proceed as in the large time case, following [4], to solve this problem: we build a solution under the form of a Lax matrix associated to the spectral curve. For brevity we will not enter the details of the resolution but only give the result, the proof is similar to the building of the parametrix away from the branch points in the large time case.

The matrix M takes the following form:

$$[M(x)]_{(j,h),(j',h')} = M_j^{(h)} \left(\xi_{j'}^{(h')}(x) \right) \quad (4-77)$$

where the functions $M_j^{(h)}$ are Baker-Akhiezer functions defined by:

$$M_0^{(0)}(x) = \frac{\prod_{(i,m)} (x - a_i^{(m)})}{\sqrt{R(x)}} \quad (4-78)$$

and

$$M_j^{(h)}(x) = c_j^{(h)} \frac{\prod_{(i,m) \neq (j,h)} (x - a_i^{(m)})}{\sqrt{R(x)}} \quad (4-79)$$

where

$$a_i^{(m)} := \xi_i^{(m)}(\infty), \quad (4-80)$$

$$R(x) := \prod_{i=1}^l (x - p_{2i-1}) (x - p_{2i}) \prod_{i,m} (x - q_i^{(m)}) (x - \tilde{q}_i^{(m)}) \quad (4-81)$$

where

$$q_i^{(m)} := \xi_0 \left(w_i^{(m)} \right) \quad \text{and} \quad \tilde{q}_i^{(m)} := \xi_0 \left(\overline{w}_i^{(m)} \right) \quad (4-82)$$

and

$$c_i^{(m)} := -i \sqrt{\frac{n_i^{(m)}}{N}}. \quad (4-83)$$

We now look at the behavior of the solution in the neighborhood of the branch points by approximating it by a local parametrix build from the Airy function.

Near the real branch points z_i , we use the local parametrices P built in the preceding section (see Eq. (3-77)). Around the imaginary branch point, we have to build a slightly

different parametrix. For this purpose, we can directly use the parametrix built in section 7 of [1]. Let us remind here this result for the parametrix around $w_i^{(m)}$.

Around the branch point $w_i^{(m)}$, let us define a local parametrix $P(x)$ for x inside a disc $\mathcal{D}_i^{(m)}(r)$ of small radius r centered in $w_i^{(m)}$. Studying the behavior of $\lambda_i^{(m)}(x) - \lambda_i^{(m-1)}(x)$ inside this disc, one can define a conformal map $f(x)$ by:

$$f(x) := \left[\frac{3}{4} \left(\lambda_i^{(m)}(x) - \lambda_i^{(m-1)}(x) \right) \right]^{\frac{2}{3}}. \quad (4-84)$$

One can remark that the neighborhood $\mathcal{D}_i^{(m)}(r)$ is mapped to the complex plane in such a way that:

$$\arg(f(x)) = \begin{cases} -\frac{2\pi}{3} & \text{for } x \in \Gamma_i^{(m)} \\ 0 & \text{for } x \in \Sigma_i \text{ in the left half-plane} \\ \frac{2\pi}{3} & \text{for } x \in \Sigma_i \text{ in the right half-plane} \end{cases} \quad (4-85)$$

With these tools, one defines the parametrix $P(x)$ in $\mathcal{D}_i^{(m)}(r)$ by:

$$P(x) := E(x) \Phi \left(N^{\frac{2}{3}} f(x) \right) D(x) \quad (4-86)$$

with $D(x)$ is a diagonal matrix defined by

$$[D(x)]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} \left[1 + \delta_{(j,h);(i,m-1)} \left(e^{\frac{N}{2}(\lambda_i^{(m-1)} - \lambda_i^{(m)})} - 1 \right) + \delta_{(j,h);(i,m)} \left(e^{\frac{N}{2}(\lambda_i^{(m)} - \lambda_i^{(m-1)})} - 1 \right) \right] \quad (4-87)$$

for $x \in \mathcal{D}_i^{(m)}(r)$, $\Phi(x)$ is the identity matrix except the 2×2 block corresponding to the intersection of the $(i, m-1)$ 'th and (i, m) 'th lines and column which takes the value

$$\begin{array}{c|cc} & (i, m-1) & (i, m) \\ \hline (i, m-1) & -y'_2 & y'_0 \\ (i, m) & -y_2 & y_0 \end{array} \quad (4-88)$$

for $0 < \arg(f(x)) < \frac{2\pi}{3}$,

$$\begin{array}{c|cc} & (i, m-1) & (i, m) \\ \hline (i, m-1) & -y'_1 & y'_0 \\ (i, m) & -y_1 & y_0 \end{array} \quad (4-89)$$

for $0 > \arg(f(x)) > -\frac{2\pi}{3}$ and

$$\begin{array}{c|cc} & (i, m-1) & (i, m) \\ \hline (i, m-1) & -y'_2 & y'_1 \\ (i, m) & -y_2 & y_1 \end{array} \quad (4-90)$$

for $\frac{2\pi}{3} < \arg(f(x)) < \frac{4\pi}{3}$ with the functions y_α defined in Eq. (3-82). The matrix E is a prefactor used to ensure that the local parametrix can be linked properly to the global parametrix on the boundary of $\mathcal{D}_i^{(m)}(r)$:

$$E(x) := M(x)L(x)^{-1} \quad \text{where} \quad L(x) := \frac{1}{2\sqrt{\pi}}\tilde{L}(x)\hat{L}(x) \quad (4-91)$$

with \tilde{L} the diagonal matrix

$$\left[\tilde{L}(x)\right]_{(j,h);(j',h')} := \delta_{(j,h);(j',h')} \left[1 + \delta_{(j,h);(i,m-1)} \left(N^{\frac{1}{6}}f(x)^{\frac{1}{4}} - 1\right) + \delta_{(j,h);(i,m)} \left(N^{-\frac{1}{6}}f(x)^{-\frac{1}{4}} - 1\right)\right] \quad (4-92)$$

and \hat{L} the identity matrix except for the 2×2 block corresponding to the intersection of the $(i, m-1)$ 'th and (i, m) 'th lines and columns:

$$\begin{array}{c|cc} & (i, m-1) & (i, m) \\ \hline (i, m-1) & i & -1 \\ (i, m) & i & 1 \end{array} \quad (4-93)$$

4.2.5 Final transformation

We now have all the ingredients in hand to perform the final transformation and build a solution to our Riemann-Hilbert problem. Let us define:

$$\begin{aligned} R(x) &:= S(x)M^{-1}(x) \quad \text{outside of the discs surrounding the branch points} \\ R(x) &:= S(x)P(x)^{-1} \quad \text{inside of the discs surrounding the branch points} \end{aligned} \quad (4-94)$$

where $P(x)$ is the local parametrix Eq. (3-77) if one considers a real branch point or Eq. (4-86) if one considers a complex branch point.

As $N \rightarrow \infty$, it is solution to the following RH problem:

- $R(x)$ is analytic inside the discs surrounding the branch and in $\mathbb{C} \setminus \left(\mathbb{R} \cup \bigcup_i \Sigma \bigcup_{(i,m)} L_i \bigcup \Gamma_i^{(m)} \right)$ outside;
- its jumps outside the discs are of the form:

$$R_+ = R - (\mathbf{I} + O(e^{-\alpha N})) \quad (4-95)$$

with $\alpha > 0$ and its jumps on the circles surrounding the branch points are such that:

$$R_+ = R - \left(\mathbf{I} + O\left(\frac{1}{N}\right) \right). \quad (4-96)$$

- as $x \rightarrow \infty$:

$$R(x) = \mathbf{I} + O\left(\frac{1}{N}\right). \quad (4-97)$$

As in the large time case this implies the behavior of R :

$$R(x) = \mathbf{I} + O\left(\frac{1}{N(|x| + 1)}\right) \quad (4-98)$$

as $N \rightarrow \infty$ (see [4]).

4.3 Asymptotics of the kernel and universality

Using this final transformation as well as the properties of the parametrices, one easily shows theorems 1.1, 1.2 and 1.3 by mimicking the proof of the large time case.

5 Perspectives

We used the Riemann-Hilbert approach to study the asymptotic behavior of N non-intersecting Brownian bridges and show the universality of their correlation functions as $N \rightarrow \infty$. Nevertheless, we did not study the critical configurations corresponding to a spectral curve where two real branch points merge. One expects to find a universal behavior given by the Pearcey kernel as it was found in the particular case studied by [5]. This critical case may be addressed by the same techniques and is left to a forthcoming work. However, as the length of these notes witnesses, the RH approach is long and requires a global information on the spectral curve to finally give local results. It would be interesting to obtain these results in a more direct way using, for example, a direct steepest-analysis as performed by Adler and Van Moerbeke [1].

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